Fundamentals of bifurcation theory and stability analysis

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Geomechanical systems are of particular interest as they involve various multiphysical, non-linear processes at several length and time scales. These complex mechanisms are described by non-linear differential equations that express the evolution of the various state variables of a system (e.g. displacements, temperature, pore pressure etc.). The solution of the governing equations, if it is possible to find, can provide complete information of the system and its behavior in time, but for specific initial and boundary conditions. Bifurcation theory and stability analysis are very useful tools for investigating qualitatively and quantitatively the behavior of complex systems without determining explicitly the solutions of its governing equations for various initial and boundary conditions. This chapter is an introduction to the corresponding mathematical theories. It aims at providing the basic ideas of bifurcation theory and stability analysis, it focuses on giving the necessary vocabulary for the classification of equilibria and of common bifurcations that are often met in applications and, finally, it presents the application of the theory for studying strain localization in solids. Some aspects related to shear band thickness, mesh dependency and generalized continua are also briefly discussed.

1 Introduction

Geomechanical systems are of particular interest as they involve various multiphysical, non-linear processes that are characterized by several length and time scales. The inherent complex geomechanical procedures span from the terrestrial kilometric scale to the nanoscale of rock porosity, grain comminution and physicochemical activity, as well as from the geological time scale to the sudden formation of shear bands related to earthquake nucleation, landslides or failure of geotechnical sites.

These complex mechanisms are described by non-linear differential equations that express the evolution of the state variables of a system in time (e.g. the evolution of displacements, temperature, pore fluid pressure, internal energy etc.). The solution of the differential equations can provide complete information of the system and its behavior in time.

Ideally we would like to compute directly and in analytical form all the solutions of a differential equation. Unfortunately, this is not possible except in the case of linear equations with constant coefficients or in the case of some special types of non-linear differential equations. Numerical methods can help us go further and by using fast computers to approximate the solutions of specific initial and boundary value problems. Nevertheless, these are unique solutions, bound to the specific choice of numerical values for the initial and boundary conditions and no further information can be deduced for the spatio-temporal evolution of the system even for small perturbations of these conditions. Moreover, a universal numerical method that can solve any problem (any system of non-linear differential equations) does not exist yet. Numerical problems such as non-convergence of the numerical algorithm and inaccurate numerical results are common in practice. Finally, in most of the cases we are not interested in the exact evolution of the complete system, but just in the evolution of some critical state variables or of its equilibrium.

It is natural therefore to ask if we can investigate the qualitative and quantitative properties of the solutions of a complex system without solving its governing equations analytically or numerically. Stability ~~analysis~~ and bifurcation analyses are the main tools for that.

A complete list of references on bifurcation and stability analysis exceeds the scope of the present chapter. Here, we refer only to some that we find fundamental from a pedagogical point of view. For an introduction to bifurcation theory and stability analysis of general dynamical systems we refer to [Bra69, Cro91, Str94]. Of course the pioneering work of Lyapunov [Lya66, Lya92a, Lya92b] is very interesting for deepening into dynamical systems and their stability. Concerning the application of bifurcation theory in solid mechanics and plasticity we refer, among others, to [Big91a, Big91b, Lem09, Ric76]. Focusing on geomechanics and multiphysical couplings for classical and generalized continua such as Cosserat, we suggest the following references [Ben00, Ben03, Vev13, Bes00, Bés01, Iss00, Per93, Ste14, Sul11, Var95, Vev12].

Despite the various theoretical and mathematical complications related to constitutive modeling, one has to bear in mind that once the equations for the (dynamical) system are established, bifurcation (and stability) analysis is a standard methodology. One needs to identify and solve for certain types of solutions that act as attractors (or repellers) of the system of equations, i.e. "special" types of solutions that irrespective of the initial data all other solutions will tend towards to (or move away from). These might be time independent *equilibria* or periodic motions, for instance. In Figure 1 we attempt to illustrate this concept. The solid lines (constant solutions) depict the time independent solutions and the dashed lines are the time dependent ones. We observe that the latter can hover around (top), deviate (middle) or approach (bottom) the equilibria as time elapses. In this way, one might know where all the solutions will tend to in time without the need of computing them. Furthermore, the bifurcation theory can help us determine whether a particular attractor of a system, which we might not even know explicitly, is the only one, under which conditions (due to parameter variations) it might lose stability and become a repeller or whether other possible solutions can exist.

For instance, consider a homogeneously deformed solid under loading. Bifurcation analysis can indicate the existence of other solutions under a given a load and their stability (for example a localized zone inside the solid, such as a shear band, can develop). In other words bifurcation theory can help us determine under which conditions a small perturbation of the reference solution (in this example the homogeneous deformation of the solid) will grow in time (unstable solution leading to strain localization) or not (see Figure 1).

It is worth emphasizing that the notion of stability, well established and defined by the original work of Lyapunov [Lya92a] in the end of 19th century, is related with the time evolution of a system. Even if in common practice time is neglected (quasi-static conditions), the transition from a state (e.g. the homogeneous deformation state) to another one (e.g. the formation of shear bands) happens in a certain time scale, which might be very short (sudden failure of brittle materials) or very slow (geological phenomena). This is why time is central in stability theory as it will be seen in the following sections.

The current chapter follows the following structure. In the beginning of section 2 we present a simple example of a dynamic mechanical system in order to introduce some basic notions of stability and bifurcation theory. Then, the necessary definitions of stable and unstable equilibria are given. The stability of general linear and non-linear systems is investigated next. In section 3 the dynamics and stability of two dimensional systems is described. A classification of the various equilibrium points is made. The dynamics can be surprisingly rich allowing even to represent Romeo's and Juliet's affair (see Love mechanics, paragraph 3.2). In section 4 we present the most common bifurcations and their classification. The notion of limit cycles is also introduced. All these sections focus on Ordinary Differential Equation's (ODE's), which is the key block for studying bifurcation and stability. The study of Partial Differential Equations (PDE's), which is of main importance in geomechanics, is based on the same principles and techniques with ODE's. In section 5 we discuss how the study of ODE's is generalized in the case of PDE's. The condition for deformation band formation (such compaction, shear, dilation bands and their combinations) is retrieved (acoustic tensor) with two different approaches and their stability is discussed. Finally, some aspects related to shear band thickness, mesh dependency and generalized continua are also briefly discussed.

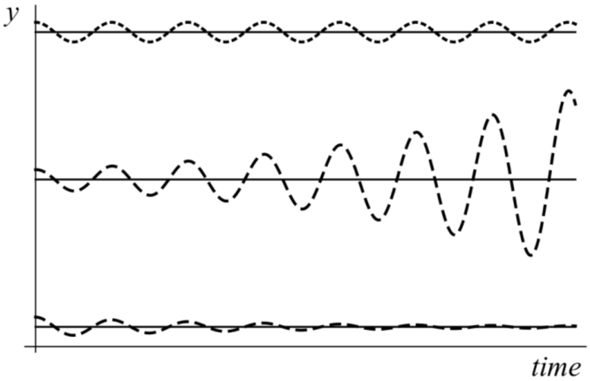
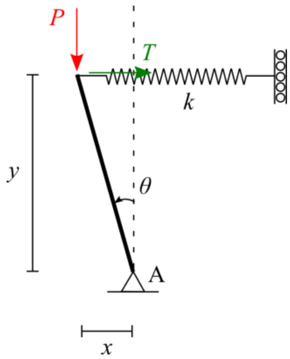


Figure 1. Different types of stability. Solid lines depict the fixed points and dashed lines depict the time evolution of solutions starting from initial conditions near them. From bottom to top the fixed points are asymptotically stable, unstable and (neutrally) stable.



$$I_A \ddot{\theta} = \Sigma M_A = P\,x - T\,y$$
$$T = k\,x$$
$$x = \ell \sin\theta$$
$$y = \ell \cos\theta$$

Figure 2: Spring - rigid beam system.

2 Bifurcation and stability of dynamical systems

Let's start with the simple example of the mechanical system of Figure 2, which is subjected to a vertical force P. The spring coefficient is k, the length of the rigid beam ℓ and its moment of inertia with respect to the out-of-plane axis passing through point A is I_A. The dynamic behavior of the system is described by the following non-linear differential equation:

$$I_A \ddot{\theta} = k\ell^2 \sin\theta \left(\frac{P}{k\ell} - \cos\theta \right) \tag{1}$$

where θ is the rotation angle around A, as shown in Figure 2. The double dot represents the second derivative in time. By setting $\dot{\theta} = \omega$ the above equation can be written in the following equivalent form:

$$\begin{cases} \dot{\theta} = \omega \\ I_{\mathrm{A}} \dot{\omega} = k\ell^2 \sin\theta \left(\dfrac{P}{k\ell} - \cos\theta \right) \end{cases} \tag{2}$$

The system is in equilibrium when $\ddot{\theta} = 0$ or equivalently when $\dot{\theta} = 0$ and $\dot{\omega} = 0$. Therefore, in order to be in equilibrium either $\cos\theta_0 = P/k\ell$ or $\theta_0 = n\pi$, where $n \in \mathbb{Z}$. Figure 3 shows all the possible values of the angle θ for which equilibrium is possible for given $P^* = P/k\ell$. Points B_i are called bifurcation points of the (equilibrium) solutions. The diagram of Figure 3 is called *bifurcation diagram* and P^* bifurcation parameter. Depending on the problem at hand various bifurcation parameters can be selected. The bifurcation diagram is a very useful tool for presenting the possible equilibria or steady states of a system (mechanical, chemical, geomechanical etc.). More details about bifurcation types and bifurcation diagrams are given in section 4. It is worth mentioning that for a given value of P^* we may have several equilibrium solutions. However, some equilibria might be stable and some other unstable, in the sense that, if we are in a certain equilibrium and a tiny perturbation takes place (a fly that sits on the beam!) the system will stay close to its initial equilibrium (stable equilibrium) or it will diverge away of it (unstable equilibrium, see Figure 1). But how stability is rigorously defined and how we can assess it for a given system?

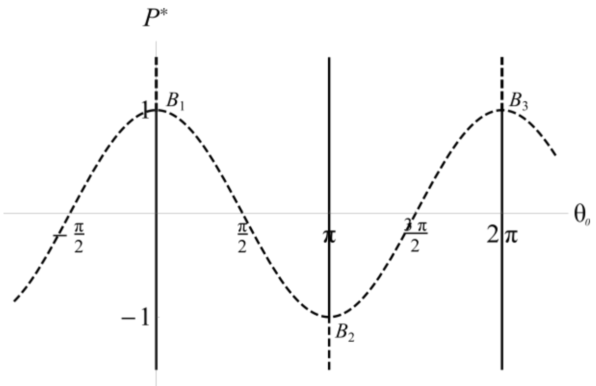


Figure 3: Bifurcation diagram. Dashed lines represent unstable braches and solid stable.

2.1 Definition of stability

Let us consider a physical system which is described by the following ODE's (set of first order ordinary differential equations):

$$\dot{\underline{y}} = \underline{f}(\underline{y}) \tag{3}$$

$\underline{y}$ is a vector of n components that contains the various quantities that determine the evolution of the physical system. The dot represents again the time derivative and $\underline{f}$ is a vector function that does not depend explicitly on the independent variable which is the time t (autonomous system). $\underline{f}$ belongs to $C^1(D)$ ($\underline{f} \in C^1(D)$), which assures existence and uniqueness of the initial value problem defined by Eq.(3). D is the n-dimensional real Euclidean space over which $\underline{f}$ is defined and C^1 denotes that $\underline{f}$ and its derivatives, in terms of the components of $\underline{y}$, are continuous. The existence and uniqueness of solutions of the initial value problem does not mean that the system has only one equilibrium point. It means that for given initial conditions the system follows a unique trajectory. In other words it can be proven that the response of the initial value problem, even if it is very sensitive to initial conditions (chaotic behavior), has a unique evolution in time as long as $\underline{f} \in C^1(D)$. Though, various equilibria points (or steady states) might exist, as shown in the previous example (Figure 2, Figure 3). In practice, when we use the term loss of uniqueness (see [Cha04] for a discussion) of solutions we refer to the existence of several different equilibrium solutions that satisfy $\underline{f}(\underline{y}_0) = 0$. $\underline{y}_0$ are called *fixed points*.

The important question, as far as applications are concerned, is if a certain equilibrium is stable or not. In other words, if at time t_0 we are in equilibrium ($\dot{\underline{y}}_0 = \underline{f}(\underline{y}_0) = 0$) and a tiny perturbation $\tilde{\underline{\psi}}$ takes place such as $\underline{\psi} = \underline{y}_0 + \tilde{\underline{\psi}}$, do we return to the initial equilibrium, $\underline{y}_0$, or the system diverges to another state? Lyapunov [Lya66, Lya92b, Lya92a] introduced the following definitions of stability:

Definition 1: The equilibrium solution $\underline{y}_0$ is said to be <u>stable</u> if for each number $\varepsilon > 0$ we can find a number $\delta > 0$ (depending on ε) such that if $\underline{\psi}(t)$ is any solution of Eq.(3) having $\left\|\underline{\psi}(t_0) - \underline{y}_0\right\| < \delta$ then the solution $\underline{\psi}(t)$ exists for all $t \geq t_0$ and $\left\|\underline{\psi}(t) - \underline{y}_0\right\| < \varepsilon$ for $t \geq t_0$ (see Figure 1,top).

$\|.\|$ denotes here the Euclidian norm ($\|\underline{a}\| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$).

Definition 2: The equilibrium solution $\underline{y}_0$ is said to be <u>asymptotically stable</u> if it is stable and if there exists a number $\delta_0 > 0$ such that if $\underline{\psi}(t)$ is any solution of Eq.(3) having $\left\|\underline{\psi}(t_0) - \underline{y}_0\right\| < \delta_0$ then $\lim_{t \to +\infty} \underline{\psi}(t) = \underline{y}_0$ (see Figure 1,bottom).

Definition 3: The equilibrium solution $\underline{y}_0$ is said to be <u>unstable</u> if it is not stable (see Figure 1,middle).

2.2 Linear systems of ODEs's

The simplest dynamical system for which stability questions can be easily addressed is the following first order linear system of ODE's:

$$\dot{\underline{y}} = \underline{\underline{A}}\,\underline{y} \tag{4}$$

where $\underline{\underline{A}}$ is a real constant $n \times n$ matrix. $\underline{y}_0 = 0$ is the equilibrium solution.

We would like to determine its evolution in time. In other words, if initially the system was in equilibrium, how a small, tiny perturbation would evolve over time? To answer this question, we can determine the general solution of the system, a task that is possible because of its linearity.

A particular solution of the above system is $\underline{\psi}(t) = \underline{\eta}\, e^{st}$, where $\underline{\eta}$ is a vector of constants with $\underline{\eta} \neq 0$. Injecting this form into Eq.(4) we obtain:

$$\left(\underline{\underline{A}} - s\underline{\underline{I}}\right)\underline{\eta} = 0 \tag{5}$$

As $\underline{\eta}$ is not the zero vector, the above equation is satisfied for s such that the determinant $\det\left(\underline{\underline{A}} - s\underline{\underline{I}}\right) = 0$. Equation (5) defines an eigenvalue problem, which has n eigenvalues, $s^{(i)}$, and n associated eigenvectors $\underline{\eta}^{(i)}$. The calculation of the determinant results to a polynomial of n degree in terms of s, which is called *characteristic polynomial* and whose roots are called *eigenvalues* $s^{(i)}$. If the eigenvalues of the system are distinct (no repeated eigenvalues, called *simple eigenvalues*) the general solution of this ODE system is:

$$\underline{y}(t) = \sum_{i=1}^{n} c_i \underline{\eta}^{(i)}\, e^{s^{(i)} t} \tag{6}$$

where c_i are constants that are determined by the initial conditions of the problem. The eigenvalues of the system can be real or imaginary. The imaginary part is responsible of an oscillatory behavior of the system while the real part is related to stability. If one of the eigenvalues is positive, then Eq.(6) indicates that the solution of the system will increase exponentially in time (monotonously increasing term).

If the characteristic polynomial has p distinct eigenvalues (roots) $s^{(i)}$ ($1 \le i \le p$) with multiplicity $m^{(i)}$ each one (if the eigenvalue k is simple, then $m^{(k)} = 1$) and associated eigenvectors $\underline{\eta}^{(i)}$, then it can be shown that the general solution of the ODE system is:

$$\underline{y}(t) = \sum_{i=1}^{p} \sum_{j=1}^{m^{(i)}} c_{i,j} \underline{\eta}^{(i)} t^{j-1} e^{s^{(i)} t} \tag{7}$$

where again $c_{i,j}$ are n, in total, constants that are determined by the initial conditions of the problem. For example, if the system consists of $n = 3$ ODE's and it has only two distinct eigenvalues (one of the eigenvalues has multiplicity 2) then its general solution is: $\underline{y} = c_{1,1}\underline{\eta}^{(1)} e^{s^{(1)}t} + c_{2,1}\underline{\eta}^{(i)} e^{s^{(2)}t} + c_{2,2}\underline{\eta}^{(i)} t e^{s^{(2)}t}$. Notice, the term $t e^{s^{(2)}t}$, which is strictly increasing in a region close to $t = 0, 0 \le t \le \varepsilon$, even if $s^{(2)} \le 0$.

By combining the aforementioned definitions of stability and the behavior of the solutions of Eq.(4) the following theorem can be proven [Bra69]:

Theorem 1:

- If all eigenvalues of $\underline{\underline{A}}$ have non-positive real parts and all those eigenvalues with zero real parts are simple, then the zero solution $\underline{y}_0 = 0$ of Eq.(4) is stable.
- If (and only if) all eigenvalues of $\underline{\underline{A}}$ have negative real parts, the zero solution of Eq.(4) is asymptotically stable.
- If one or more eigenvalues of $\underline{\underline{A}}$ have a positive real part, the zero solution of Eq.(4) is unstable.

In other words the stability of the equilibrium state of a linear system is investigated by simply studying the eigenvalues of the matrix $\underline{\underline{A}}$.

Can this theorem be extended for non-linear systems as the one presented in the beginning of this section?

2.3 Non-linear systems of ODE's

The system described by Eq.(3), $\dot{\underline{y}} = \underline{f}(\underline{y})$, is non-linear in the sense that $\underline{f}(\underline{y})$ is a non-linear function of $\underline{y}$. Expressing its solution $\underline{\psi}(t)$ in the form:

$$\underline{\psi}(t) = \underline{y}_0 + \underline{\tilde{\psi}}(t) \tag{8}$$

where $\underline{y}_0$ is one of the equilibrium solutions (fixed point), we obtain:

$$\dot{\underline{\tilde{\psi}}}(t) = \underline{f}(\underline{y}_0 + \underline{\tilde{\psi}}(t)) = \underline{f}(\underline{y}_0 + \underline{\tilde{\psi}}(t)) - \underline{f}(\underline{y}_0) \tag{9}$$

If the difference at the right hand side can be written in the following almost-linear form:

$$\dot{\underline{\tilde{\psi}}}(t) = \underline{\underline{A}}\,\underline{\tilde{\psi}} + \underline{p}(\underline{\tilde{\psi}}) \tag{10}$$

where $\underline{\underline{A}} = \underline{\underline{J}}(\underline{y}_0) = \left\{ \left. \frac{\partial f_i}{\partial y_j} \right|_{\underline{y}=\underline{y}_0} \right\}$ the Jacobian of $\underline{f}(\underline{y})$ at point $\underline{y}_0$ ($\underline{\underline{A}}$ is a real constant $n \times n$ matrix), $\underline{p}$ a continuous function with $\underline{p}(\underline{0}) = \underline{0}$ and $\lim_{\underline{\psi} \to 0} \frac{\left\| \underline{p}(\underline{\tilde{\psi}}) \right\|}{\left\| \underline{\tilde{\psi}} \right\|} = 0$, then the following theorem can be proven [Bra69]:

Theorem 2: Suppose that $\underline{p}$ is continuous, $\left\| \underline{\tilde{\psi}} \right\| < k$, where $k > 0$ is a constant, and $\underline{p}$ is small in the sense that $\lim_{\underline{\psi} \to 0} \frac{\left\| \underline{p}(\underline{\tilde{\psi}}) \right\|}{\left\| \underline{\tilde{\psi}} \right\|} = 0$, then:

- If all eigenvalues of $\underline{\underline{A}}$ have negative real parts, the solution $\underline{\tilde{\psi}} = 0$ of Eq.(10) is asymptotically stable.
- If one or more eigenvalues of $\underline{\underline{A}}$ have a positive real part, the solution $\underline{\tilde{\psi}} = 0$ of Eq.(10) is unstable.

Notice that if the second derivative of $\underline{f}$ with respect to $\underline{y}$ exists, then the term $\underline{p}(\underline{\tilde{\psi}})$ is the remainder of a Taylor expansion of $\underline{f}$, which satisfies $\underline{p}(\underline{0}) = \underline{0}$ and $\lim_{\underline{\psi} \to 0} \frac{\left\| \underline{p}(\underline{\tilde{\psi}}) \right\|}{\left\| \underline{\tilde{\psi}} \right\|} = 0$. If all eigenvalues of $\underline{\underline{A}}$ have non-positive real parts and there exists at least one eigenvalue with zero real part then the dynamics of the linearized system do not represent the dynamics of the non-linear system and no conclusion can be safely derived for the stability of the non-linear system. However, in the special case of conservative (systems where a conserved quantity exists, e.g. the total energy) or reversible systems (systems with time reversal symmetry) it can be proven that when all the eigenvalues of $\underline{\underline{A}}$ have non-positive real parts and there exists at least one eigenvalue with zero real part, then all orbits close to a fixed point are closed (see [Str94]). In this case the (isolated) fixed point is called non-linear center and is stable in the Lyapunov sense (but not asymptotically stable).

The above theorem gives the conditions for which any perturbation $\underline{\tilde{\psi}}$ is bounded, decays or grows exponentially with time. According to the definitions of stability, the system will be respectively (*asymptotically*) *stable* or *unstable*. Therefore, the eigenvalues of the matrix $\underline{\underline{A}}$ can provide useful information about the stability of an equilibrium solution, even in the case of non-linear ODE's. The investigation of stability by using the above theorem is called Linear Stability Analysis (LSA), as it is based on the linearization of $\underline{f}(\underline{y})$.

2.4 An example of Linear Stability Analysis

The system presented in the beginning of this section (Eq.(2)) is expressed in the form of Eq.(3) as follows:

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} \text{ and } \underline{f} = \begin{bmatrix} \omega \\ \frac{k\ell^2}{I_{\mathrm{A}}} \sin\theta \left(\frac{P}{k\ell} - \cos\theta \right) \end{bmatrix} \tag{11}$$

At equilibrium $\underline{\dot{y}} = 0$ and $\underline{y} = \underline{y}_0$. Perturbing the equilibrium solution we replace $\underline{y}(t)$ by $\underline{\psi}(t) = \underline{y}_0 + \underline{\tilde{\psi}}(t)$ (Eq.(8)). Performing a Taylor expansion of $\underline{f}$ up to the first order around the point $\underline{y} = \underline{y}_0$ we retrieve a linear equation of the form of (10) where: $\underline{\underline{A}} = \underline{\underline{J}}(\underline{y}_0) = \left\{ \left. \frac{\partial f_i}{\partial y_j} \right|_{\underline{y}=\underline{y}_0} \right\} = \begin{bmatrix} 0 & 1 \\ \frac{k\ell^2}{I_\mathrm{A}}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \frac{k\ell^2}{I_\mathrm{A}}\sin^2\theta_0 & 0 \end{bmatrix}$.

The characteristic polynomial of the eigenvalue problem is:

$$s^2 - \frac{k\ell^2}{I_\mathrm{A}}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 - \frac{k\ell^2}{I_\mathrm{A}}\sin^2\theta_0 = 0 \tag{12}$$

which leads to two eigenvalues:

$$s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_\mathrm{A}}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \frac{k\ell^2}{I_\mathrm{A}}\sin^2\theta_0} \tag{13}$$

Now we can investigate the stability of the various branches of the bifurcation diagram (Figure 3). When we are on the sinusoidal branch $\frac{P}{k\ell} - \cos\theta_0 = 0$, and therefore $s_{1,2} = \pm\left|\sin\theta_0\right|\sqrt{\frac{k\ell^2}{I_\mathrm{A}}}$, which means that there is always a positive eigenvalue (imaginary part is zero). When we are on the vertical branches $\sin\theta_0 = 0$ and $s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_\mathrm{A}}\left(\frac{P}{k\ell} - 1\right)}$ for $\theta_0 = 2n\pi$ or $s_{1,2} = \pm\sqrt{-\frac{k\ell^2}{I_\mathrm{A}}\left(\frac{P}{k\ell} + 1\right)}$ for $\theta_0 = (2n+1)\pi$. In the first case, i.e. for $\theta_0 = 2n\pi$, if $P > k\ell$ then one of the eigenvalues is positive (the imaginary part is zero), which means that the system is unstable. If $P < k\ell$ then $s_{1,2} = \pm\mathbf{i}\sqrt{\frac{k\ell^2}{I_\mathrm{A}}\left|\frac{P}{k\ell} - 1\right|}$, which are two distinct imaginary eigenvalues ($\mathbf{i} = \sqrt{-1}$) and consequently, according to paragraph 2.3, the equilibrium is (neutrally) stable (conservative and reversible system). In the second case, i.e. for $\theta_0 = (2n+1)\pi$, if $P < -k\ell$ then one of the eigenvalues is positive (the imaginary part is zero), which means that the system is unstable. If $P > -k\ell$ then $s_{1,2} = \pm\mathbf{i}\sqrt{\frac{k\ell^2}{I_\mathrm{A}}\left|\frac{P}{k\ell} + 1\right|}$ which are two distinct imaginary eigenvalues and consequently, according to paragraph 2.3. the equilibrium is (neutrally) stable. Figure 3 summarizes these results in the bifurcation diagram. If $P = \pm k\ell$ and $\theta_0 = n\pi$, then $s_{1,2} = 0$ and no conclusion can be drawn about the stability of these points.

3 Stability of two dimensional linear dynamical systems

The general form of a two dimensional linear system is:

$$\begin{aligned} \dot{y}_1 &= a\, y_1 + b\, y_2 \\ \dot{y}_2 &= c\, y_1 + d\, y_2 \end{aligned} \tag{14}$$

The equilibrium solution of this system (fixed point) is obviously $\underline{y}_0 = 0$. The constants matrix is $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the characteristic polynomial:

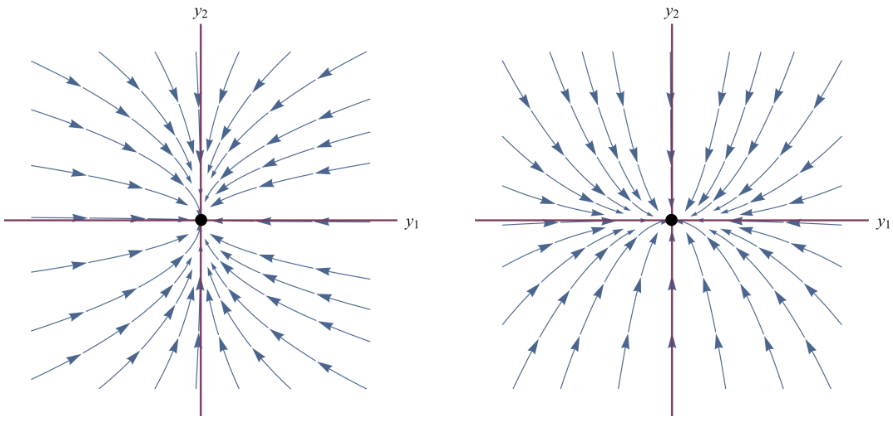
$$s^2 - \tau s + \Delta = 0 \tag{15}$$

where $\tau = a + d$ and $\Delta = ad - bc$. Let s_1 and s_2 be the roots of the characteristic polynomial (eigenvalues of $\underline{\underline{A}}$):

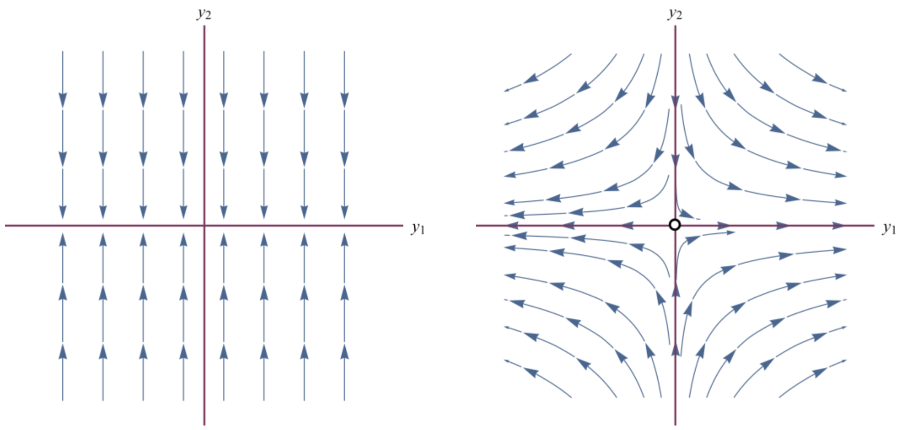
$$s_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad s_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}, \tag{16}$$

3.1 Classification of fixed points

Take for instance the following case for $\underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvalues are $s_1 = a$ and $s_2 = -1$. Let also a real. The solution of this linear system is $y_1 = c_1 e^{at}$, $y_2 = c_2 e^{-t}$ (see paragraph 2.2). The initial conditions determine the constants c_1 and c_2 . Plotting this solution in the (phase) space (y_1, y_2) we obtain the trajectories presented in Figure 4 for various initial conditions. Such a diagram is called phase diagram and it depicts the dynamics or the so-called mathematical flow of the system. From the solution of this system we get $\frac{dy_1}{dy_2} = -a \frac{c_1}{c_2} e^{(a-1)t}$, which illustrates that the dynamic behavior of the system evolves (is concentrated) in the direction with the slowest in absolute value eigenvalue (slow eigen-direction of the linearized system). In other words, for $a<-1$, y_2 reduces faster than y_1 and the solution approaches the equilibrium point having as an asymptote the axis y_2 (see Figure 4a). The direction of the slowest evolution of the system is called *slow manifold* (slow manifold of an equilibrium point of a dynamical system). The contrary holds for $-1<a<0$ (see Figure 4b). In both cases the equilibrium point is an *attractor* (y_1 and y_2 are stable manifolds) and the fixed point is called *stable node*. If both eigenvalues are real and positive then the fixed point is called *unstable node*. If $a=0$ then y_2 is constant ($y_1=c_1$) and the system evolves as shown in Figure 4c. In the case that $a>0$ the system is unstable and the equilibrium point is a *saddle node* (Figure 4d). For initial conditions such that $c_1=0$ the system will evolve towards the equilibrium point. However, for tiny values of c_1 the system will diverge from the equilibrium point (y_1 is a stable manifold and y_2 unstable - saddle).



(a) (b)

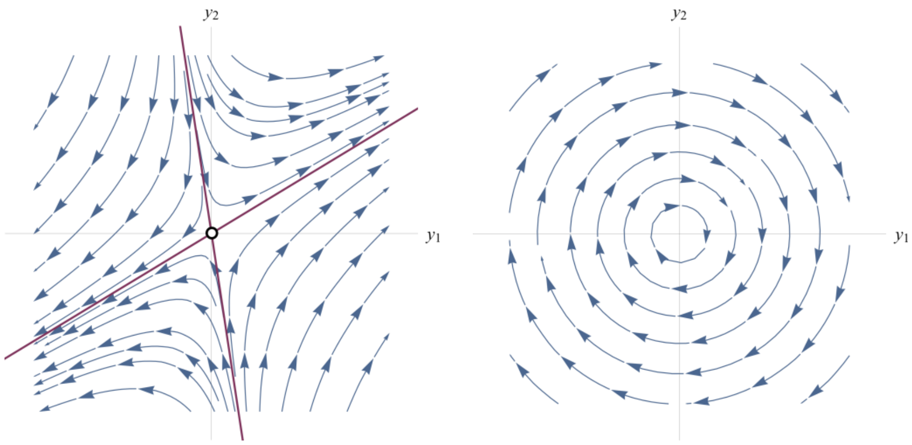


(c) (d)

Figure 4. Phase diagram for (a) $a < -1$, (b) $-1 < a < 0$, (c) $a = 0$ and (d) $a > 0$. A black dot indicates a stable fixed point (stable node, attractor), while an open circle indicates an unstable one (unstable node-repeller or saddle).

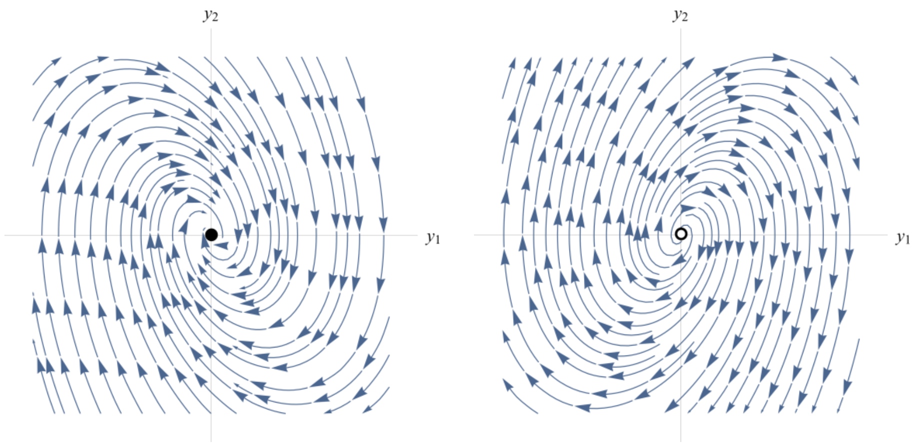
In the general case (see Figure 5) of real eigenvalues the manifolds (eigenvectors) are not perpendicular (except if $\underline{\underline{A}}$ is symmetric). If the real part of both eigenvalues of $\underline{\underline{A}}$ is zero and they have different non-zero imaginary parts the equilibrium is neutrally stable (stable in the Lyapunov sense for a linear system, see definitions in paragraph 2.2 and 2.3). If the eigenvalues are complex with negative real part then we have oscillations of reducing amplitude until equilibrium (Figure 6). If the real part is positive and the imaginary part non-zero then we diverge from equilibrium (oscillations with increasing amplitude, see also Figure 1). In the case of repeated but non-null eigenvalues, if the eigenvectors are distinct, then we have a star node, and when there is only one eigenvector the fixed point is called degenerated node (Figure 7). For a nice online application for generating phase diagrams for various $\underline{\underline{A}}$ we refer to [Che16].

Figure 8 summarizes the various types of fixed points in function of τ and Δ. In the bifurcation example of the previous section $\tau = 0$. Therefore, we had either saddle points (unstable equilibria) or centers (neutrally stable equilibria).



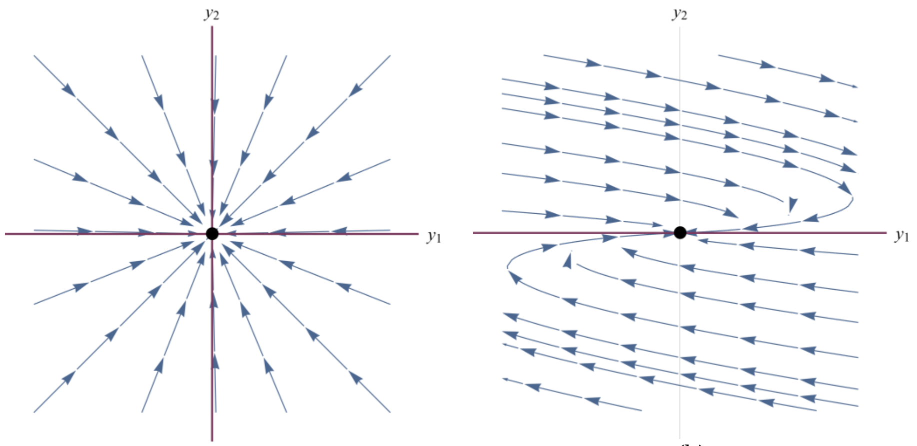
(a) (b)

Figure 5. (a) Unstable fixed point with non-orthogonal eigenvectors (saddle). (b) Neutrally stable fixed point (center).



(a) (b)

Figure 6.Stable (a) and unstable (b) spiral fixed points.



(a) (b)

Figure 7. Degenerate cases: (a) star node and (b) degenerate node.

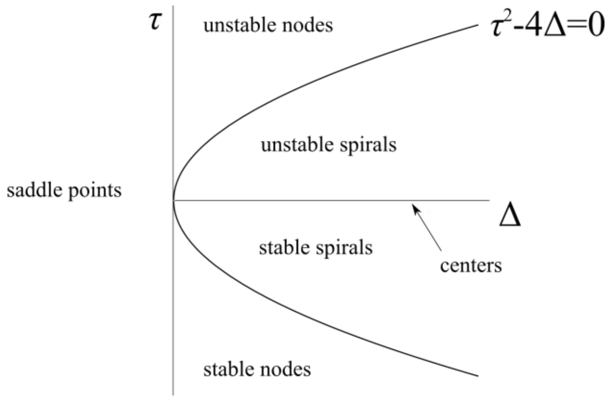


Figure 8. Classification of fixed points of a two dimensional dynamical systems.

3.2 Love mechanics: Romeo and Juliet

Two dimensional systems are certainly more interesting than one dimensional. While one dimensional systems can have nodes that are either stable or unstable and the solution might simply diverge or converge towards the equilibrium points, two dimensional systems involve richer dynamics, such as oscillations.

Strogatz (see [Spr04, Str88, Str94]) used a simple linear two dimensional system to describe the romantic affair between Romeo and Juliet! In his example y_1 describes the love of Romeo for Juliet (R) and y_2 the love of Juliet for Romeo (J):

$$\dot{R} = a\,R + b\,J$$
$$\dot{J} = c\,R + d\,J \tag{17}$$

Positive values for J or R signify love and negative hate. $J = R = 0$ signifies mutual indifference. The parameters a, b, c, d have a simple meaning as well. $a < 0$ means that Romeo is a cautious lover. The more he realizes that he loves Juliet the more he is afraid, which reduces his affection. But if $b > 0$ is great enough the affection of Juliet makes his feelings stronger ($\dot{R} > 0$). Similarly, Juliet's love is characterized by the parameters c, d .

Let's investigate the scenario with $a = d = 0$, $b > 0$ and $c < 0$. Let's say that at the beginning Romeo is in love with Juliet and also Juliet with him. Though, the more Romeo loves her ($R\uparrow$) the more Juliet is afraid and wants to run away ($\dot{J} = c\,R < 0 \Rightarrow J < 0$). Romeo gets disappointed and backs off ($\dot{R} = b\,J < 0 \Rightarrow R < 0$). But then Juliet begins to find him again attractive ($\dot{J} = c\,R > 0 \Rightarrow J > 0$) and after a while Romeo echoes her ($\dot{R} = b\,J > 0 \Rightarrow R > 0$). But then again Juliet loses her interest and so on… The sad outcome of this relation is a never ending cycle of love and hate (see Figure 5b)! If Romeo was also a cautious lover ($a < 0$) then their love would end soon following a stable spiral towards the fixed point of mutual indifference $J = R = 0$. If Romeo was an enthusiastic lover ($a > 0$) their relation would be an unstable spiral of increasing hate and love, but again if $a > 0$ is very high the more Romeo loves Juliet the more she is afraid and draws away. However, if both Romeo and Juliet were attracted in the beginning, their love would become a love fest provided that their love characteristics a, b, c, d were such that to have an unstable node or saddle in the phase space (see Figure 5b). In other words, love is instability!

Of course in reality the dynamics of love are much more complicated than this simple model and certainly non-linear. Non-linearities and more degrees of freedom were taken into account in [Spr04] and interesting dynamics were observed.

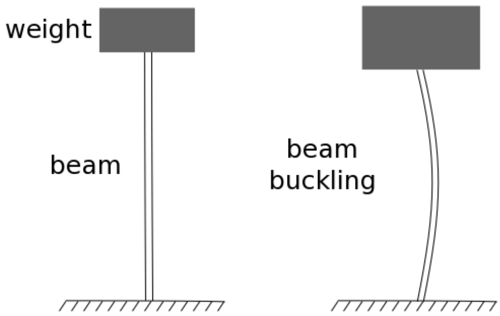


Figure 9. Sketch of beam buckling due to high load. Here, the load acts as a control (bifurcation) parameter, which determines the transition from axial deformation to a buckled, flexural state.

4 Common types of bifurcations

The evolution laws that we commonly use in Mechanics (and other scientific disciplines) can involve parameters that are unknown or non-constant (e.g. the loading of a beam or the elastic parameter of a spring). The dynamics of a system can significantly change with the variations of these parameters. For instance, in the example of section 2, the fixed point $\theta_0 = 0$ becomes unstable for values of the normalized vertical load $P^* > 1$ (see Figure 3). This is typical in many mechanical systems and central for the design of structures (see buckling of a beam due to high load, Figure 9).

In general, depending on the values of the parameters, fixed points can be created (appear) or destroyed (disappear) and/or their stability can change. These qualitative changes in the static and dynamic response of a system of equations are called *bifurcations* and the parameter values at which they first occur are called *bifurcation points*. The study of bifurcations is important since it provides the onset of instabilities and the transition across different states depending on the variation of the governing (bifurcation) parameters. In the following we will go through some of the most well-known (mathematical) bifurcation types for ODE's.

4.1 Saddle-node bifurcation

The most fundamental bifurcation is the appearance and disappearance of equilibrium points for different values of a bifurcation parameter. This is the so called *saddle-node* (or *fold*) *bifurcation* of equilibria. In this case, as the parameter varies, two fixed points of the underlying system of equations move towards each other, collide and mutually annihilate. The following differential equation is a classic example of this kind of bifurcation:

$$\dot{y} = \mu + y^2 \quad (18)$$

where μ is a real number that can admit any real value. When μ is negative, the right-hand-side of equation (18) (equilibrium solution) has two fixed points, one stable and one unstable (see Figure 10).

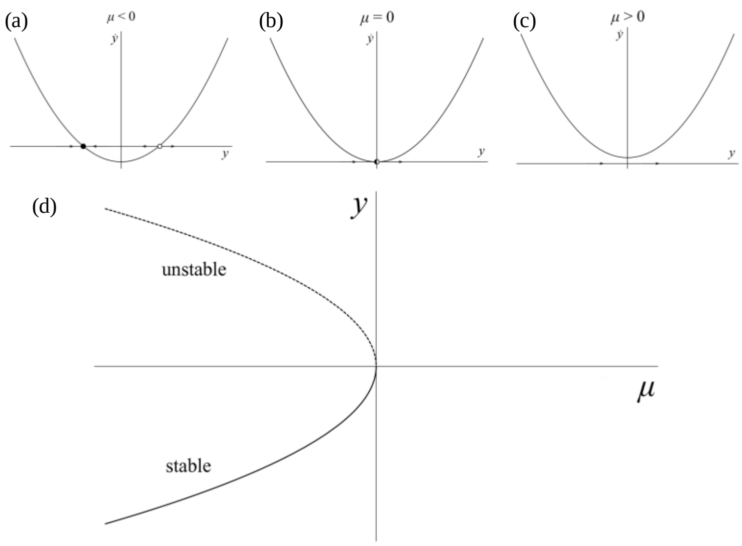


Figure 10. Saddle-node bifurcation. A half full circle denotes half stable fixed points.

As $\mu \to 0$ from negative values, the two fixed points move towards each other and they collide to a half-stable point at $y_0 = 0$ for $\mu = 0$. This is the bifurcation point, since for $\mu > 0$ the dynamics change completely, equation (18) presents no fixed points and predicts infinite growth of y in time. This behavior is depicted in Figure 10a-c. As μ plays the role of the independent variable, we can plot the steady state solutions of equation (18) in an $\mu - y_0$ diagram (Figure 10d), where we can observe the number of steady state solutions in function of the parameter μ, as well as their stability (calculated as described in section 2). As mentioned in section 2, such a diagram, where the fixed points of the equations (for a norm of the solution) are plotted against the bifurcation parameter is called *bifurcation diagram*.

Note that equation (18) along with its symmetric $\dot{y} = \mu - y^2$, are representative of all saddle-node bifurcations. This means that close to a saddle bifurcation, the dynamics of a given system are qualitatively the same with $\dot{y} = \mu - y^2$ or $\dot{y} = \mu + y^2$ (see

Appendix). Equations like equation (18), which can characterize the dynamics of any system near a bifurcation point are called *normal forms* of that bifurcation.

4.2 Transcritical bifurcation

The transcritical bifurcation happens when a pair of fixed points exchange stability as the bifurcation parameter varies. Its normal form is:

$$\dot{y} = \mu y - y^2 \tag{19}$$

As shown in Figure 11a-c, the point $y_0 = 0$ is always a fixed point. Starting from negative values of μ, $y_0 = 0$ is stable and there exists a second fixed point, $y_0 = \mu$, which is unstable. As the value of the parameter μ increases, the second fixed point moves towards $y_0 = 0$ and for $\mu = 0$ (which is also the bifurcation point in this case) they collapse on a half-stable point. Upon further increasing of the parameter value, the two fixed points reappear but have opposite stability, $y_0 = 0$ is unstable and $y_0 = \mu$ is stable. Thus, we can say that the transcritical bifurcation is a mechanism for exchanging stability between two fixed points. This is more apparent if one observes the corresponding bifurcation diagram (Figure 11d).

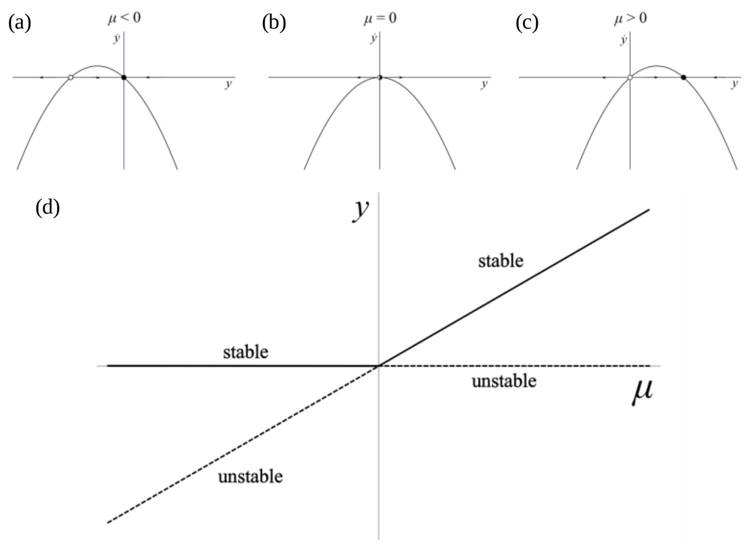


Figure 11. Transcritical bifurcation.

4.3 Supercritical and subcritical pitchfork bifurcation

Pitchfork bifurcation (both the super- and the sub-critical one) is common in problems that have symmetry and describe the appearance (or disappearance) of a symmetrical pair of fixed points after some critical value of the bifurcation parameter.

Revisiting the example of the buckling of a beam (see Figure 2 or Figure 9), after the load exceeds the critical threshold there exists no preferred direction for the deformation and only a defect in the mechanical problem or the perturbation itself can lead the beam to “choose” one direction or another, thus breaking its symmetry.

The normal form of the supercritical pitchfork bifurcation is:

$$\dot{y} = \mu y - y^3 \tag{20}$$

Notice that changing the variable $y \to -y$ does not change the equation of the system. This symmetry justifies mathematically the aforementioned existence of a symmetrical pair of fixed points.

The fixed point $y_0 = 0$ exists for all $\mu \in \mathbb{R}$. For $\mu \le 0$ it is stable. At $\mu = 0$ the pitchfork bifurcation occurs and for $\mu > 0$ a symmetric pair of stable fixed points appears ($y_0 = \pm\sqrt{\mu}$). Therefore, three fixed points exist ($y_0 = \pm\sqrt{\mu}$ and $y_0 = 0$) for $\mu > 0$.

The reason for which this type of bifurcation is called “pitchfork” becomes apparent upon observing the bifurcation diagram of Figure 12d.

The subcritical pitchfork bifurcation has the following normal form:

$$\dot{y} = \mu y + y^3 \tag{21}$$

As shown in Figure 13, the corresponding bifurcation diagram is similar to the one of Figure 12d, but the pitchfork is inverted. The pair $y_0 = \pm\sqrt{-\mu}$ is unstable and it exists only for $\mu < 0$. In addition, even though the origin is a fixed point for all $\mu \in \mathbb{R}$, it is stable only for $\mu < 0$. For all $\mu > 0$ there is no stable equilibrium solution and the system *blows up*, i.e. $y \to \pm\infty$ as time elapses. Furthermore, one can show that the blow-up happens in finite time for all initial conditions. It is worth mentioning that this type of bifurcation is the normal form of the bifurcation of the example studied in sections 2 and 3 (see Figure 2 and Figure 3).

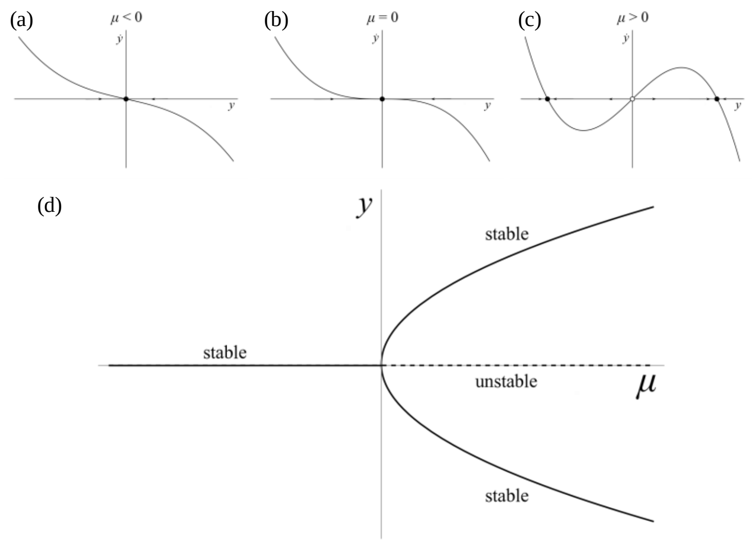


Figure 12. Supercritical pitchfork bifurcation.

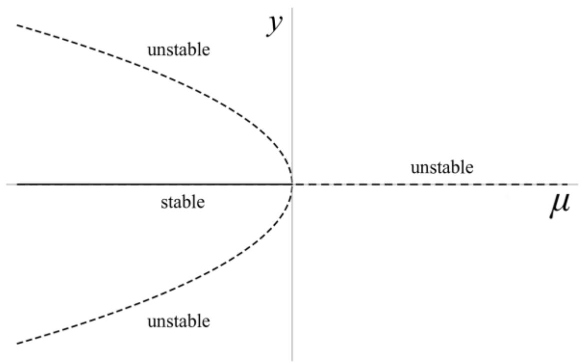


Figure 13. Bifurcation diagram of the subcritical pitchfork bifurcation.

4.4 From one to two dimensions - Limit cycles

All the cases considered in the previous paragraphs concerned bifurcations of equilibrium solutions in one-dimensional problems. What about bifurcations in problems of high order?

Non-linear dynamical systems of order higher than one can present perfectly periodic solutions. Such solutions appear on the phase space as isolated closed orbits, which can attract or repel all neighboring trajectories, much like the fixed points.

These orbits are called *limit cycles*. Limit cycles are an inherent phenomenon of two or higher dimensional systems that are non-linear. Even though, linear systems can present closed orbits, when the fixed point is a stable center (neutral stability, see Figure 5b), such solutions are non-isolated, i.e. if $x(t)$ is a periodic solution, then $c\,x(t)$ is also a periodic solution for all $c \in \mathbb{R}^*$.

An illustrative example of a system with a stable limit cycle in polar coordinates is:

$$\begin{aligned} \dot{r} &= r\left(1-r^2\right) \\ \dot{\theta} &= 1 \end{aligned} \tag{22}$$

where $r \geq 0$. It is easy to identify that the two equations are uncoupled and that the first one if treated alone, it has two fixed points, namely $r = 0$ (unstable) and $r = 1$ (stable). This means that all trajectories approach $r = 1$. However, the system of two equations has no fixed points at all because $\dot{\theta} = 1 \neq 0$. $\dot{\theta} = 1$ describes the angular velocity, which is constant. Therefore, all trajectories on the phase plane are approaching the unit circle ($r = 1$) monotonically. This can be visualized if we revert again to Cartesian coordinates (Figure 14 (a)), i.e. $x(t) = r(t)\cos\left(\theta(t)\right)$ and $y(t) = r(t)\sin\left(\theta(t)\right)$. The evolution in time of the x-coordinate, for $r(0) = 0.01$ and $\theta(0) = 0$ is presented in Figure 14 (b). As we can see, the amplitude of the oscillations is $r = 1$ and the period is $T = 2\pi$.

One of the most famous examples of equations that present limit cycles is the *van der Pol equation*,

$$\ddot{y} + \mu\left(y^2 - 1\right)\dot{y} + y = 0 \tag{23}$$

where $\mu \geq 0$ is a parameter. In this equation, the non-linear term $\mu\left(y^2 - 1\right)\dot{y}$ forces the oscillation. The limit cycle is no longer a circle (Figure 15a) and the waveform is not sinusoidal (Figure 15b).

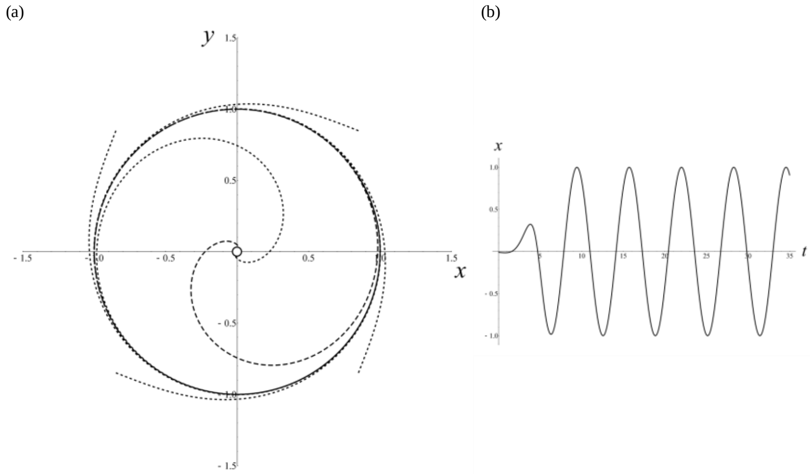


Figure 14. (a) Phase diagram of the system of Eq.22. We observe the trajectories moving towards the limit cycle of the system. (b) Evolution in time of the system of Eq.22 for $r(0) = 0.01$ and $\theta(0) = 0$.

A question that follows naturally is if fixed points and limit cycles are the only possible attractors (or repellers) of the trajectories of a system of ODE's. The answer to that is negative for higher dimensions. In two dimensions, the dimensionality of the system (and thus the corresponding trajectories on the phase space) is equal to the dimensionality of the limit cycles (both equal to two) and hence all trajectories on the phase space can be attracted to either points or closed orbits. On the contrary, dynamical systems of order $n \geq 3$ can have trajectories that might be in an open, bounded domain, yet, they can move freely inside it without settling into a fixed point or a closed orbit. They can be attracted to topological manifolds (called *stable manifolds*) or even to complex geometric objects that are called *strange attractors* or *fractals*. The study of such complex (or chaotic) dynamics is out of the scope of the present chapter and the reader should refer to [Hal91, Str94] for a first introduction to these phenomena.

4.5 Bifurcations in two dimensions - Supercritical and subcritical Hopf bifurcation

We are now ready to answer the question about bifurcations in two dimensional systems. In terms of bifurcations of fixed points, all the basic examples discussed in paragraphs 4.1-4.3 have their analogs in two (and in higher) dimensions. The corresponding normal forms in two dimensions are:

$$\begin{aligned}\dot{y}_1 &= \mu + y_1^2 \\ \dot{y}_2 &= -y_2\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{y}_1 &= \mu y_1 - y_1^2 \\ \dot{y}_2 &= -y_2\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{y}_1 &= \mu y_1 - y_1^3 \\ \dot{y}_2 &= -y_2\end{aligned} \tag{26}$$

$$\begin{aligned}\dot{y}_1 &= \mu y_1 + y_1^3 \\ \dot{y}_2 &= -y_2\end{aligned} \tag{27}$$

for the saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork respectively. It is easy to prove that at the bifurcation point $(\mu = 0)$, the corresponding linearized problem has a zero eigenvalue. This means that they always involve the collision of fixed points. Furthermore, irrespectively of the dimensionality of the problem, these types of bifurcations are inherently one-dimensional phenomena in the sense that they occur on the one-dimensional unstable manifold of the unstable fixed point. There exists however another way for a fixed point to lose stability and it involves the creation or destruction of a limit cycle around it.

This case is the so-called *Hopf* (or *Andropov-Hopf*) bifurcation. Let us assume that the dynamical system at hand, $\dot{y} = f(y, \mu)$, has a stable fixed point. This means that the eigenvalues, λ_1, λ_2, of the Jacobian matrix of the system have negative real parts. The imaginary part is not necessarily zero. In other words the eigenvalues lie on the left half-plane of the complex plane $\mathbb{C}$ (see Figure 16). For a two dimensional system, there are only two possible cases for its eigenvalues, either $\lambda_1, \lambda_2 \in \mathbb{R}^-$ or they are complex conjugates. Let us then assume that there exists a value of the parameter $\mu = \mu_H$ for which the fixed point loses stability. In the first case, as we approach this bifurcation point by varying the parameter μ one of the eigenvalues becomes zero. This corresponds to the cases of saddle-node, transcritical and pitchfork bifurcations. In the second case, the pair of complex conjugate eigenvalues crosses simultaneously the imaginary axis into the right half-plane (Figure 16). The latter is the fundamental mechanism described by the Hopf bifurcation.

As mentioned before, the Hopf bifurcation describes the creation or destruction of a limit cycle around a fixed point when the latter loses stability. The first potential scenario is the creation of a limit cycle from a fixed point. In this case, for all values $\mu < \mu_H$, the system is stable and the fixed point is a stable spiral (Figure 6). As μ increases it approaches and then surpasses the critical value μ_H for which the spiral becomes unstable. This is the case of the supercritical Hopf bifurcation and its normal form is given by:

$$\begin{aligned} \dot{x} &= \mu x - y - x\left(x^2 + y^2\right) \\ \dot{y} &= x + \mu y - y\left(x^2 + y^2\right) \end{aligned} \tag{28}$$

or equivalently in polar coordinates:

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= 1 \end{aligned} \tag{29}$$

It is worth mentioning that the unstable spiral is surrounded by a stable limit cycle. Notice that the system (29) is just a generalization of the system 22. For $\mu < 0$ it yields that $\dot{r} < 0$ and thus all oscillations have decreasing amplitude. This means that the only attractor is the origin and it is a stable spiral (Figure 17a). For $\mu = 0$ the origin becomes a center. For $\mu > 0$ as shown also in paragraph 4.4 the origin becomes an unstable spiral and is surrounded by a stable limit cycle (Figure 17b). If we consider $x(t) = r(t)\cos\left(\theta(t)\right)$ it is easy to show from the roots of $\mu r - r^3$ that the amplitude of the oscillations is $r = \sqrt{\mu}$ and the period $T = 2\pi$.

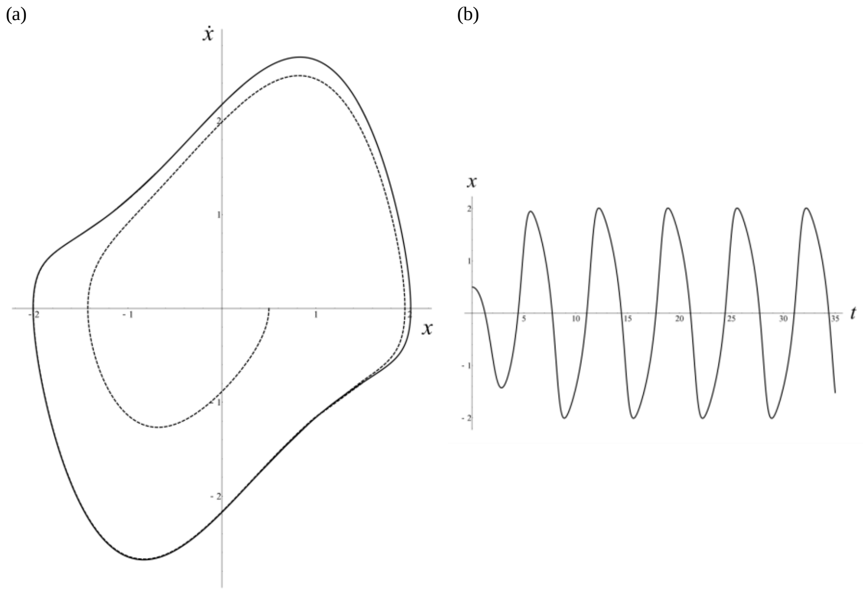


Figure 15. (a) Phase diagram of the van der Pol equation for $\mu = 1$. (b) Evolution of the solution of the van der Pol equation for $y(0) = 0.5$ and $\mu = 1$.

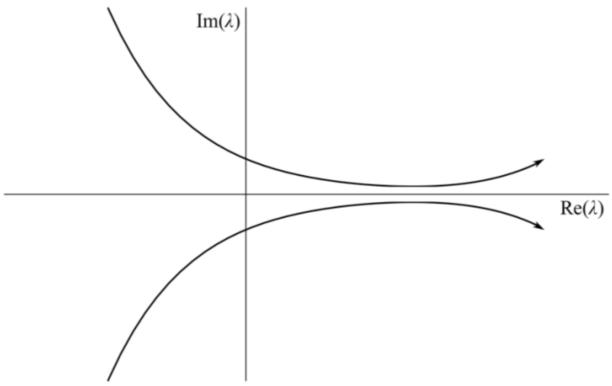


Figure 16. Sketch of a pair of complex eigenvalues crossing the imaginary axis.

We should notice here that the normal form represents the so-called *topological equivalent* of the Hopf bifurcation. This means that all limit cycles that are created by a Hopf bifurcation are equivalent (in a mathematical sense) to an oscillation of amplitude $\sqrt{\mu}$ and period 2π (in other words, of angular velocity $\omega = 1$). Based on the system 29 one can construct more general systems of equations that can admit different modes of sinusoidal wave forms as solutions. Such a system is:

$$\dot{r} = \mu r - ar^3$$
$$\dot{\theta} = \omega + br^2 \tag{30}$$

where ω is the frequency of the infinitesimal oscillations (near $\mu = 0$) and b describes the dependency of the frequency (and of the angular velocity) on the amplitude. For $a > 0$ it can be shown that the amplitude is $r = \sqrt{\mu / a}$ and the period is $T = 2\pi / \left(\omega + br^2\right)$.

The second potential scenario is the destruction of a limit cycle and it us called *subcritical Hopf bifurcation*. Its normal form is as follows:

$$\dot{x} = \mu x - y + x\left(x^2 + y^2\right)$$
$$\dot{y} = x + \mu y + y\left(x^2 + y^2\right) \tag{31}$$

or equivalently in polar coordinates:

$$\dot{r} = \mu r + r^3$$
$$\dot{\theta} = 1 \tag{32}$$

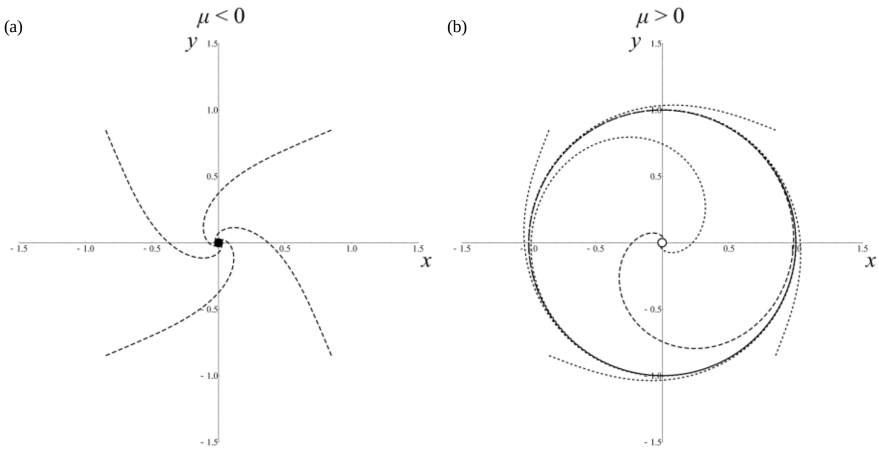


Figure 17. Phase diagram of the supercritical Hopf bifurcation. We observe the transition from a stable spiral in Figure (a) to an unstable one which is surrounded by a stable limit cycle (b).

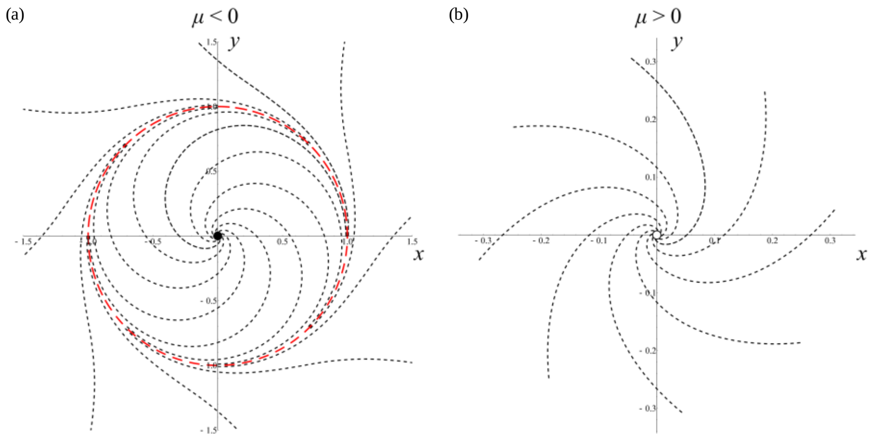


Figure 18. Phase diagram of the subcritical Hopf bifurcation. Figure (a) is for $\mu < 0$ where we observe that origin is a stable spiral that is surrounded by an unstable limit cycle (marked with red dashed line). All trajectories starting from inside the cycle tend to the origin while those starting out of it diverge. The cycle radius decreases with increasing μ until it collapses to the fixed point for $\mu = 0$. Figure (b) is for $\mu > 0$ where we observe that there exists only the origin as a fixed point and it is an unstable spiral.

In this case, for $\mu < 0$ the right-hand-side of the radial equation has two roots, $r_0 = 0$ and $r_0 = \mu$. It is easy to show that the origin is a stable fixed point (stable spiral). However, the second root represents an unstable limit cycle for the system. This means that all trajectories that start inside the cycle move towards the origin whereas all the trajectories that start outside the cycle diverge since the cycle repels them (Figure 18a).

As the value of the parameter increases the radius of the cycle decreases and collapses to the origin for $\mu = 0$. For $\mu > 0$ the limit cycle is destroyed and the origin exchanges stability with it becoming unstable (Figure 18b). This means that for $\mu > 0$ there is no stable solution (fixed or periodic) for the system (32).

4.6 Mathematical Bifurcations in PDE's

The bifurcations presented in the previous paragraphs are just indicative cases. As stated in paragraph 4.4 the higher the order of the system, the more complex the behavior can be. However, change in the number of the equilibrium solutions as well as changes in the stability of equilibrium or periodic solutions are the most common and important bifurcations that can occur in systems of equations modelling problems in Mechanics. Even though the analysis was so far restricted in ODE's these mathematical bifurcations are observed in PDE's as well.

One famous example is the Bratu equation [Bra14, Gel63] here written as boundary value problem with symmetric boundary conditions and in one dimension:

$$\begin{aligned} 0 &= \frac{d^2T(x)}{dx^2} + \lambda e^{T(x)} \\ T(-1) &= T(1) = 1 \end{aligned} \tag{33}$$

where x is the spatial coordinate and λ a bifurcation parameter. This equation can describe the time-independent behavior of an infinite layer under simple shear [Che89]. This means that it provides the steady state solutions ($\frac{\partial T(x,t)}{\partial t} = 0$), which is the equivalent of the fixed points of ODE's. The steady state problem of equations (33) has two solutions for $\lambda < \lambda_c$, one for $\lambda = \lambda_c$ and none for all $\lambda > \lambda_c$, where λ_c is the is the critical value of λ for which the bifurcation occurs. This type of bifurcation is a saddle-node bifurcation.

Notice that when dealing with PDE's, the equivalent of a fixed point is a time-independent solution which can either be homogenous (i.e. constant in space) or inhomogeneous (i.e. non-constant profile in space). One way of studying the stability of equilibrium solutions of PDE's (and their bifurcations to an extent) is using the so-called Linear Stability Analysis. An example of that method will be presented in paragraph 5.2. Such an analysis can predict the growth or decay of perturbations near an equilibrium solution, thus providing information about the stability of the equilibrium solution. By that means, one can derive when one steady state is preferred from another and determine the bifurcation point from the onset of instability. A famous example is the formation of Bénard (convection) cells in the Rayleigh-Bénard problem (Figure 19).

In this problem, there are various types of steady states and corresponding bifurcations. The critical parameter that governs these instabilities is called Rayleigh number and for low values, where conduction prevails, a linear temperature profile is predicted. Upon reaching a critical value though, a bifurcation occurs and convection becomes the dominant mechanism. In this case the flow appears to be steady in time but periodic in space (Figure 19b). For more information, the reader should refer to [Cha61] where the results of linear stability analyses for various boundary value problems of this type are presented.

In the following we will present the study of a problem that is more relevant to Geomechanics applications, the localization of shear and volumetric strain

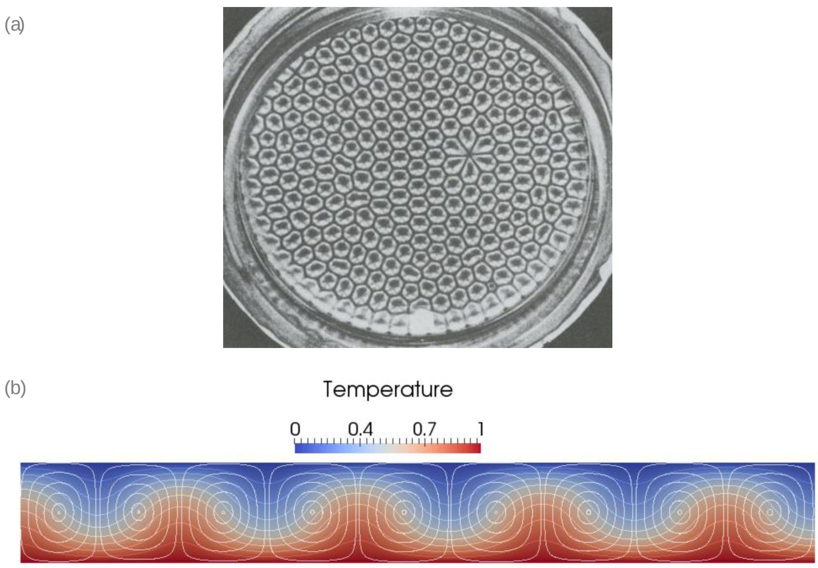


Figure 19. (a) Experiments on hydrothermal convection where the famous Bénard cells appear. (b) Streamlines and isotherms in numerical simulation of the corresponding two dimensional problem.

5 From ODE's to PDE's

The above sections were dedicated to the notion of stability and bifurcation focusing on ODE's. In this section we try to extend the above concepts to Partial Differential Equations (PDE's) who cover a variety of systems including mechanics of solids, in general, and geomechanics. A classical problem of bifurcation and instability is strain localization in materials. Strain localization is frequently manifested as thin bands where deformation is localized. Depending on the kinematics of strain localization three main types of deformation bands are distinguished. These are dilation (or extension) bands, shear bands and compaction bands [Ber02]. Whatever their type is, deformation bands appear at the moment that the homogeneous deformation of a system becomes an unstable equilibrium solution. In other words the system bifurcates to a non-homogeneous solution where the strain is localized. The classical approach for determining when this localization takes place is based on calculating the determinant of the acoustic tensor [Rud75].

5.1 Deformation bands and the acoustic tensor

Consider a homogeneous, homogeneously deformed solid subjected to quasi-static increments of deformation. Let's assume that after an increment, a deformation band is formed, which breaks the aforementioned homogeneity of the deformation field (and consequently of the stress field) as shown in Figure 20. The displacement field remains continuous across the boundaries of the band, but its gradient does not (different strains inside the band):

$$\Delta u_i = 0 \quad \text{and} \quad [\![\Delta u_{i,j}]\!] = g_i n_j \tag{34}$$

where [[.]] denotes discontinuity across the deformation band boundary (e.g. $[[a]] = a^+ - a^-$), n_i is the orientation vector of the deformation band with $i = 1,2,3$ is the three-dimensional space, u_i the displacement field and Δ denotes the increment of a field. $(\ )_{,i}$ denotes derivation in terms of x_i .

The vector g_i describes the direction of the discontinuity and its inner product with the orientation of the band, n_i , determines the type of the deformation band (see Figure 20). In particular, if $n_i g_i = 0$, the deformation band is a pure shear band, if $n_i g_i = -1$ a pure compaction band and if $n_i g_i = +1$ a pure dilation (extension) band. The intermediate states, $0 < n_i g_i < 1$ and $-1 < n_i g_i < 0$ correspond respectively to dilatant and contracting shear bands.

In quasi-static conditions, the stress vector has to be continuous across the deformation band boundary:

$$\Delta t_i = [\![\Delta\sigma_{ij}]\!] n_j = 0 \tag{35}$$

Consider the class of materials that for a small increment Δ, the constitutive law can be written (linearized) as follows:

$$\Delta\sigma_{ij} = L_{ijkl} \Delta u_{k,l} \tag{36}$$

The tensor L_{ijkl} can be continuous across the boundary of the band ($[\![C_{ijkl}]\!] = 0$) or discontinuous in the sense that elastic unloading can occur outside the band, while continued inelastic loading continues within the band. In the first case we say that we have *continuous bifurcation*, while in the second *discontinuous bifurcation*. It is shown that continuous bifurcation precedes discontinuous bifurcation [Ric80].

By replacing Eq.(36) into (35) and using (34) we obtain:

$$[\![\Delta\sigma_{ij}]\!] n_j = L_{ijkl} [\![u_{k,l}]\!] n_j = n_j L_{ijkl} n_l \, g_k = 0 \tag{37}$$

The tensor $\Gamma_{ik} = n_j L_{ijkl} n_l$ is called *acoustic tensor*. If its determinant is not zero, then the g_k has to be zero, which means that the deformation is continuous along the assumed deformation band. In other words no discontinuity of the gradient of the displacement field can appear across the boundary of the deformation band and the homogeneous solution prevails. Otherwise, if:

$$\left|\Gamma_{ik}\right| = 0 \tag{38}$$

the homogeneous solution seizes to be the only one and deformation bands are possible. For an orientation n_i the type of the deformation band is given by the (eigenvector g_i).

The above condition for strain localization is independent of the material constitutive behavior as long as Eq.(36) can be written. For instance, for an elastoplastic material whose plastic behavior is a function of the first and second invariants of the stress tensor (Figure 21), Issen and Rudnicki [Iss00] (see also [Bes00]) showed that under axisymmetric compression conditions of loading strain localization is possible when the hardening modulus becomes smaller than a critical value h_{cr} for given values of μ and β (see Figure 22).

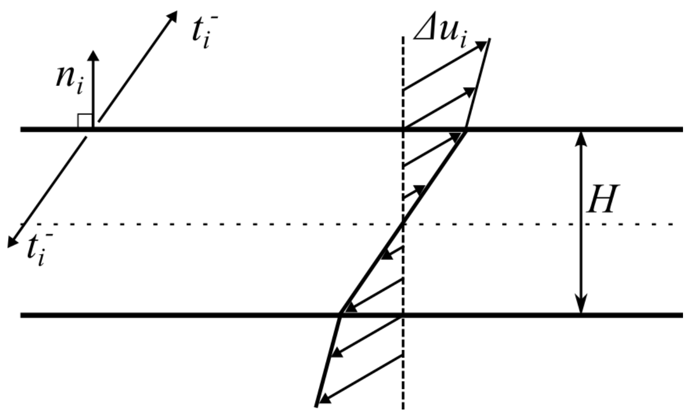


Figure 20. Schematic representation of a deformation band and of the discontinuity of the displacement field.

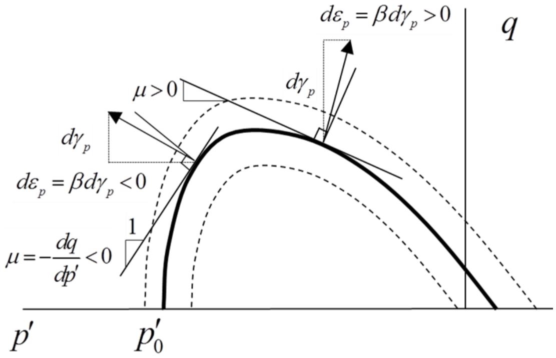


Figure 21. Elastoplastic yield envelope with hardening/softening (dotted lines). Compression is considered negative.

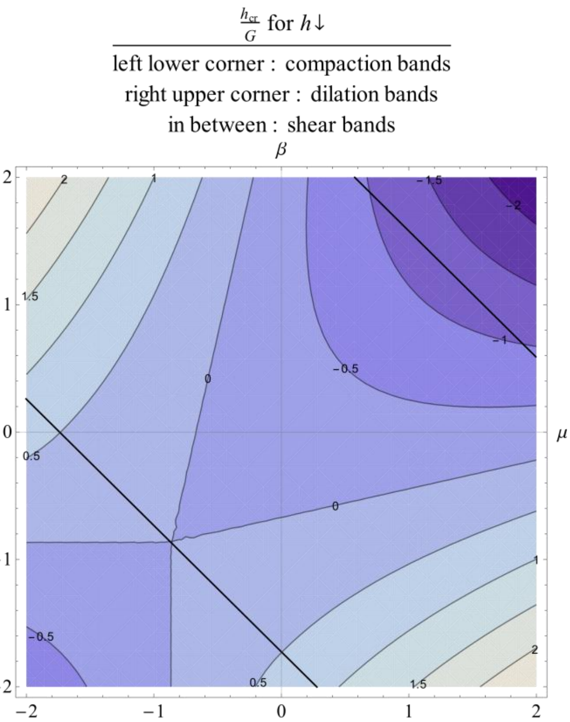


Figure 22. Critical hardening values in function of the β and μ for strain localization [Iss00]. Notice that for non-associate plastic flow rule, localization can occur even with hardening (h_{cr} >0). The above diagram was derived by using the acoustic tensor criterion for localization (Eq.(38)).

For the above derivations we considered quasi-static conditions. If we remove this restriction, the jump of the shear stresses at the boundary of the shear band is not necessarily zero due to acceleration (not in equilibrium). From the linear momentum balance we obtain:

$$\Delta t_i = [\![\Delta\sigma_{ij}]\!] n_j = -\rho c \; \gamma_i \tag{39}$$

where c is the velocity of a propagating discontinuity in direction n_i such that $[[\gamma_i]] = [[\Delta v_i]] = -c g_i$ (see Hadamard conditions on propagating discontinuities [Had03, Lem09]). Inserting Eq.(36) into (39) and using (34) we get:

$$\left(n_j L_{ijkl} n_l - \rho c^2 \delta_{ik} \right) g_k = 0 \tag{40}$$

This equation shows that if there are accelerating waves ($[[\gamma_i]] = -c g_i \neq 0$) the eigenvalues of the tensor Γ_{ij} are equal to the square root of their wave velocity c^2. This is why Γ_{ij} is called *acoustic tensor*. The condition of localization derived in quasi-static conditions (Eq.(38)) corresponds to $c = 0$ or in other words to the existence of stationary acceleration waves.

5.2 Deformation bands as an instability problem

The same condition with Eq.(38) can be derived by studying the stability of the homogeneous solution of the continuous system. In this case we do not have a system of ODE's anymore for which we saw how to investigate the stability of an equilibrium point, but a PDE. Nevertheless, the bifurcation analysis approach is similar. Stability is defined as in paragraph 2.1.

The general PDE's of the problem are:

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{41}$$

where ρ is the density of the material and the double dot represents the second time derivative (acceleration). Suppose again a homogeneous, homogeneously deformed solid that it is in equilibrium:

$$\sigma^*_{ij,j} = 0 \tag{42}$$

Considering the same class of materials that can be linearized around the above equilibrium point such as Eq.(36) to hold we get:

$$\sigma_{ij} = \sigma_{ij}^* + \Delta\sigma_{ij} = \sigma_{ij}^* + L_{ijkl}\Delta u_{k,l} \tag{43}$$

where, as in 5.1 Δu_i is the increment in displacements, i.e. $\Delta u_i = u_i - u_i^*$. Δu_i can be seen also as a perturbation of the reference, homogeneous solution u_i^* (see paragraph 2.1). By injecting the above equation in (41) and using (42) we obtain:

$$L_{ijkl}\Delta u_{k,lj} = \rho \Delta\ddot{u}_i \tag{44}$$

Notice that L_{ijkl} is calculated at u_i^* and therefore it is independent of Δu_i (Taylor expansion of σ_{ij} around ε_{ij}^*). Therefore, Eq.(44) is a linear PDE that can be solved by separation of variables (or Fourier transform). The linearization of the stress tensor around the equilibrium point (Eq.(36)) is central in strain localization analysis. Consequently, the conditions derived either in paragraph 5.1 or in the current one are valid as far as this linearization is possible.

Using the method of separation of variables, $\Delta u_i = X(x_k)U_i(t)$. Replacing in (44) we obtain:

$$L_{ijkl} X_{,lj}\, U_k(t) = \rho X\, \ddot{U}_i \tag{45}$$

This equation has sinusoidal solutions in terms of X . Moreover, we are looking for deformation bands, which are planar as shown in Figure 20. Therefore, the solution in terms of X takes the form: $X(x_i) = e^{i k n_i x_i}$,where n_i is the orientation vector of the deformation band as in the previous paragraph and k the wave number (of the perturbation). If λ is the wavelength corresponding to the wave number k ($k = 2\pi/\lambda$), in order to satisfy the boundary conditions at the boundary of the deformation band $\lambda = H/N$, where N is a integer. Inserting X in Eq.(45) and by setting $\dot{U}_i = V_i$ we obtain the following the following system of ordinary differential equations:

$$\begin{aligned} \dot{V}_i &= -\frac{1}{\rho}\left(\frac{2\pi}{\lambda}\right)^2 n_j L_{ijkl} n_l\, U_k \\ \dot{U}_i &= V_i \end{aligned} \tag{46}$$

In this way we transformed the PDE's of the problem to a system of ODE's, which we can study in the same way as in the previous sections. As shown in section 2, the above equations take solutions of the form $U_k(t) = g_k e^{st}$. After some algebraic manipulations the eigenvalue problem becomes:

$$\left[n_j L_{ijkl} n_l + \rho \left(\frac{\lambda s}{2\pi} \right)^2 \delta_{ik} \right] g_k = 0 \tag{47}$$

which is identical to (40) by setting $c = i\dfrac{\lambda s}{2\pi}$. Notice that $\Delta u_i = U_i e^{i\frac{2\pi}{\lambda} n_p x_p + st}$ $= e^{i\frac{2\pi}{\lambda}\left(n_p x_p - i\frac{s\lambda}{2\pi} t \right)} = e^{i\frac{2\pi}{\lambda}\left(n_p x_p - ct \right)}$, which describes a wave travelling with speed c. If the real part of s is positive then the homogeneous solution u_i^* is unstable and the system bifurcates to a non-uniform solution, a band, with direction n_i. As before, the type of the deformation band (compaction, shear, dilation band) is determined by the product $n_i g_i$. It is worth emphasizing that the above condition is independent of the specific constitutive law, provided that it is rate-independent. For rate dependent materials, a similar approach can be followed. The methodology is quite general and can be applied in many problems, including problems with multiphysical couplings, such as thermo-poro-chemo-mechanical couplings (e.g. [Ste14, Sul15]). Moreover, even though a Cauchy (Boltzmann) continuum was considered here, the same approach can be applied in Cosserat or even higher order continua (e.g. [Müh88, Sul11]).

If the eigenvalues of Γ_{ij} do not depend on the (perturbation) wavelength λ and s, then the acceleration wave velocity c is constant (does not depend on λ). If in addition they have a positive real part, the perturbation that propagates faster has zero wave length because for $\lambda \to 0$, $s \to \infty$. In other words the minor imperfection in size will propagate faster and dominate the other imperfections of larger wavelength. This is why in the classical Cauchy continuum, which has no internal lengths, the deformation band thickness is zero (the localization takes place on a mathematical plane). The fact that the smallest perturbation propagates faster justifies also the mesh dependency in Finite Element calculations. For instance, in the frame of classical simulations in elastoplasticity of Cauchy rate-independent continua with softening behavior (or even in perfect plasticity), the numerically predicted shear band thickness depends on the finite element discretization and on the element size (Figure 23).

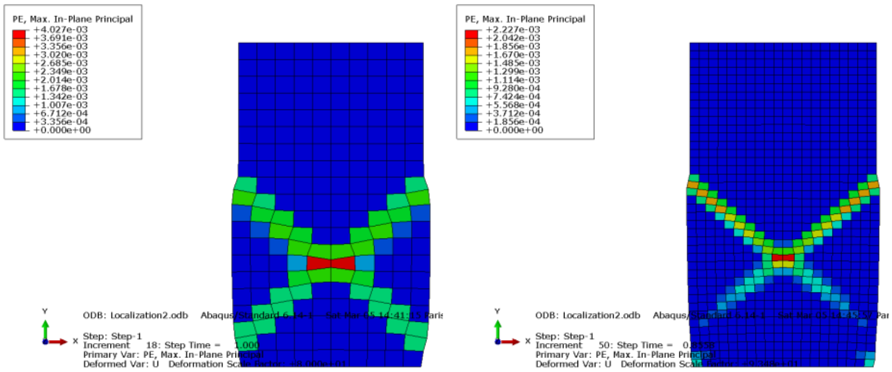


Figure 23. Shear band formation and mesh dependency for a rate-independent elastoplastic, von Mises, Cauchy medium with strain softening. The shear band thickness is always 1-2 elements thick and therefore mesh dependent. The plastic strains and the global energy dissipation are also mesh dependent. Abaqus v6.14 was used for the simulations.

6 Summary

The target of the present chapter was to give the basic ideas and tools of bifurcation theory and stability analysis. The definition of (Lyapunov) stability was given, as well as the fundamental theorems that allow studying the stability of linear and nonlinear systems of ODE's. The notion of bifurcation was explained and illustrated through examples and a classification of the most common bifurcations and instabilities was presented. The focus was given on ODE's as their behavior is central for understanding bifurcation and stability. The study of PDE's is an extension of the ideas presented for ODE's and it was presented in the last section. The strain localization conditions of homogeneously deformed solids were derived as an example (acoustic tensor). After studying this chapter the reader would be able to distinguish the basic notions of stability and bifurcation and apply the different concepts in more complicated systems in geomechanics that are characterized of advanced constitutive law and multiphysical couplings.

Appendix

Let $\dot{y} = f\left(y,\mu\right)$ be a dynamical system with a bifurcation point at $y = y_0$ for $\mu = \mu_c$. A Taylor expansion of the equation yields:

$$\dot{x} = f\left(y_0, \mu_c\right) + \left(y - y_0\right) \left.\frac{\partial f}{\partial y}\right|_{\left(y_0, \mu_c\right)} +$$
$$\left(\mu - \mu_c\right) \left.\frac{\partial f}{\partial \mu}\right|_{\left(x_0, \mu_c\right)} + \frac{1}{2}\left(y - y_0\right)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(y_0, \mu_c\right)} + \ldots \tag{48}$$

In the case of saddle-node bifurcations we have that the term $f\left(y_0, \mu_c\right) = 0$ since y_0 is a fixed point and $\left.\partial f / \partial y\right|_{\left(y_0, \mu_c\right)} = 0$ by definition for this specific bifurcation. Therefore,

$$\dot{y} = a\left(\mu - \mu_c\right) + b\left(y - y_0\right)^2 + \ldots \tag{49}$$

where, $a = \left.\partial f / \partial \mu\right|_{\left(y_0, \mu_c\right)}$ and $b = 1/2 \left.\partial^2 f / \partial y^2\right|_{\left(y_0, \mu_c\right)}$. Thus, for $\left(y, \mu\right)$ sufficiently close to $\left(y_0, \mu_c\right)$ along with $a, b \neq 0$, we can neglect the higher order terms resulting to the normal forms: $\dot{y} = \mu - y^2$ or $\dot{y} = \mu + y^2$.

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