ALERT Geomaterials
Doctoral School 2018

# The Principle of Virtual Powers definition and use

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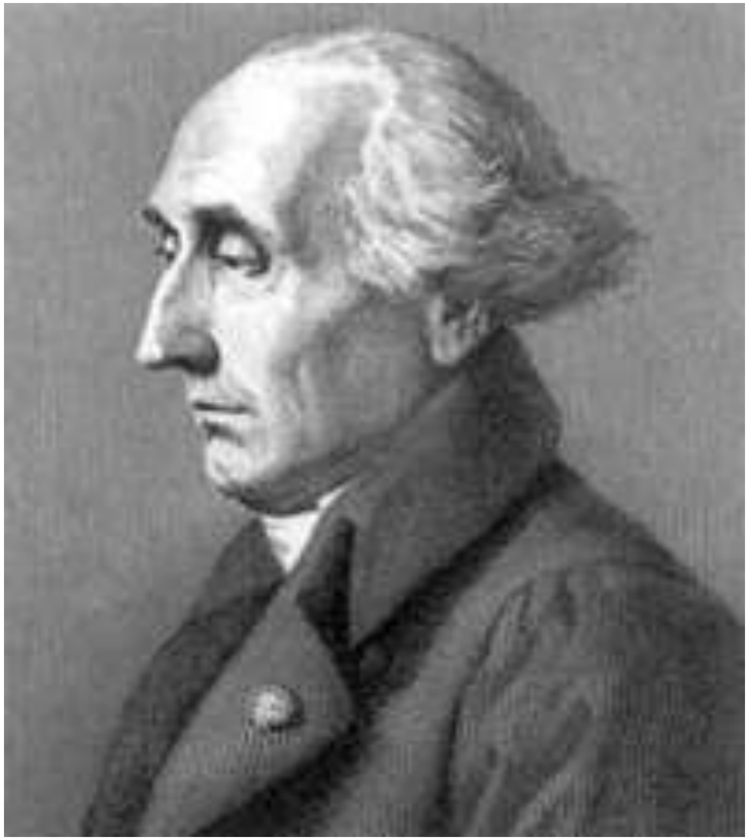
# Objectives

- Understand the Principle of Virtual Powers
- Use it in simple problems of statics
- Use it in continuum mechanics
- Use it as a starting point for Finite Element formulations

# Prerequisites

- Tensor calculus
- Patience and a nice problem to solve

# The Principle of Virtual Powers



*Giuseppe Luigi Lagrangia, 1736-1813*
*(Joseph-Louis Lagrange)*

# MÉCHANIQUE

## ANALITIQUE;

*Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.*

*A PARIS,*

Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

*AVEC APPROBATION ET PRIVILEGE DU ROI.*

**1788**

# MÉCHANIQUE ANALITIQUE.

## PREMIERE PARTIE.

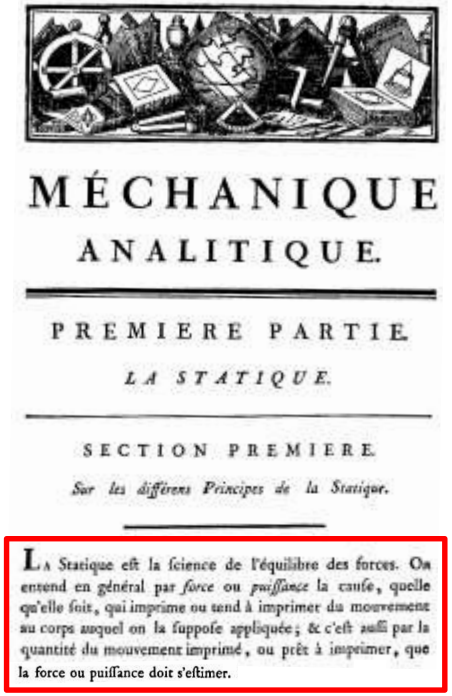
## *LA STATIQUE.*

### SECTION PREMIERE.

*Sur les différens Principes de la Statique.*

LA Statique eſt la ſcience de l'équilibre des forces. On entend en général par *force* ou *puiſſançe* la cauſe, quelle qu'elle ſoit, qui imprime ou tend à imprimer du mouvement au corps auquel on la ſuppoſe appliquée; & c'eſt auſſi par la quantité du mouvement imprimé, ou prêt à imprimer, que la force ou puiſſance doit s'eſtimer.

- **Statics** is the science of the **equilibrium of forces**
- With ***force*** (or ***power***) we mean the ***cause of mouvement***
- It is through the quantity of mouvement that the force can be quantified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# The Principle of Virtual Velocities (or Powers)

*Si un ſyſtême quelconque de tant de corps ou points que l'on veut tirés, chacun par des puiſſances quelconques, eſt en équilibre, & qu'on donne à ce ſyſtême un petit mouvement quelconque, en vertu duquel chaque point parcoure un eſpace infiniment petit qui exprimera ſa viteſſe virtuelle; la ſomme des puiſſances, multipliées chacune par l'eſpace que le point où elle eſt appliquée, parcourt ſuivant la direction de cette même puiſſance, ſera toujours égale à zero, en regardant comme poſitifs les petits eſpaces parcourus dans le ſens des puiſſances, & comme négatifs les eſpaces parcourus dans un ſens oppoſé.*

# Original statement of PVP in English

**If** [any system of bodies or points as we want, is acted upon by any system of forces,]
**is in equilibrium, and we give to this system any small motion,**
**then** [by virtue of the fact that each point travels an infinitesimally small space that expresses its **virtual velocity**,]
**the sum of each force multiplied by the space that the point** [where it is applied] **travels** [along the direction of the same power,]
**will always be equal to zero**, [regarding as positive the small distances followed in the direction of the powers and as negative those travelled in the opposite direction.]

# Make it simple…

**The body will be in equilibrium**

***if, and only if,***

**the power generated by the forces acting on it, is zero**

under **any possible** **(= virtual) velocity** of the body.

# Lagrange was far ahead of his time…

A long time ago, in a galaxy far, far away….

[courtesy: The last Jedi, film, 2018]

# The scientific problem:

**BAD** $\tilde{v}$ **GOOD**

$F_B$

$F_G$

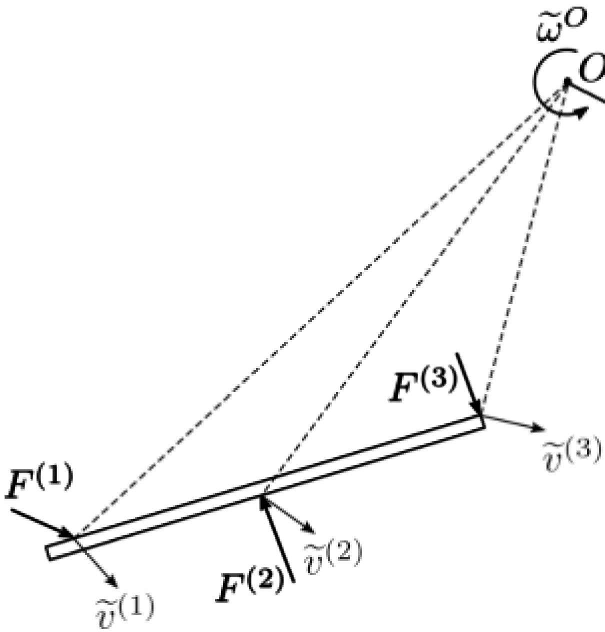
$$\tilde{\mathcal{P}} = 0, \forall \tilde{v}$$

$$-F_B \tilde{v} + F_G \tilde{v} = 0, \forall \tilde{v}$$

$$(-F_B + F_G)\tilde{v} = 0, \forall \tilde{v}$$

$$F_B = F_G$$

# PVP applied to a single body




$$\text{equilibrium} \iff \sum_i \widetilde{\mathcal{P}}^{(i)} = 0, \forall \widetilde{\boldsymbol{v}}^{(i)}$$

$$\sum_i \boldsymbol{F}^{(i)} \cdot \widetilde{\boldsymbol{v}}^{(i)} = 0, \forall \widetilde{\boldsymbol{v}}^{(i)}$$

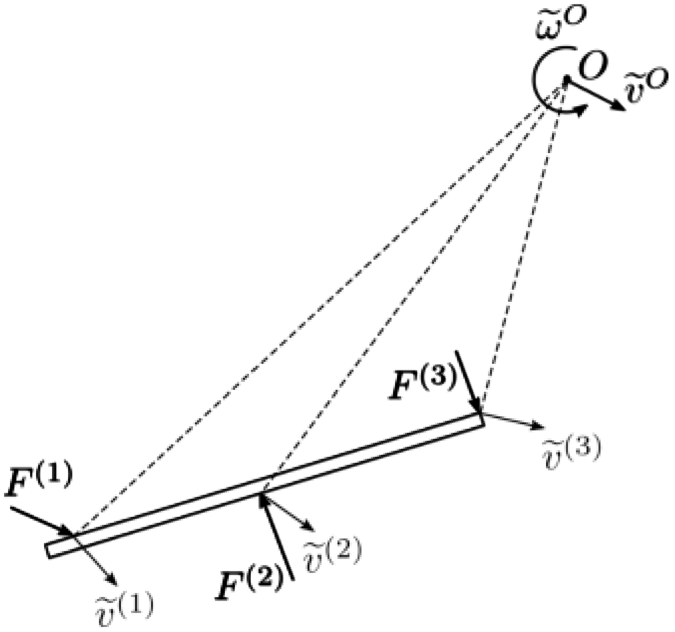
Rigid body:

$$\widetilde{\boldsymbol{v}}^{(i)} = \widetilde{\boldsymbol{v}}^O + \widetilde{\boldsymbol{\omega}}^O \times \boldsymbol{r}^{(O,i)}$$

$$\left(\sum_i \boldsymbol{F}^{(i)}\right) \cdot \widetilde{\boldsymbol{v}}^O + \left(\sum_i \boldsymbol{r}^{(O,i)} \times \boldsymbol{F}^{(i)}\right) \cdot \widetilde{\boldsymbol{\omega}}^O = 0, \forall \widetilde{\boldsymbol{v}}^O, \widetilde{\boldsymbol{\omega}}^O$$

$$\forall \widetilde{\boldsymbol{v}}^O, \widetilde{\boldsymbol{\omega}}^O \implies \begin{array}{c} \sum_i \boldsymbol{F}^{(i)} = 0 \\ \sum_i \boldsymbol{M}^{(O,i)} = 0 \end{array}$$

# PVP and ‘Newton’ equivalence




$$\text{equilibrium} \iff \begin{array}{c} \sum_i \boldsymbol{F}^{(i)} = 0 \\ \sum_i \boldsymbol{M}^{(O,i)} = 0 \end{array}$$

multiplying by any $\widetilde{\boldsymbol{v}}^O, \widetilde{\boldsymbol{\omega}}^O$ and adding:

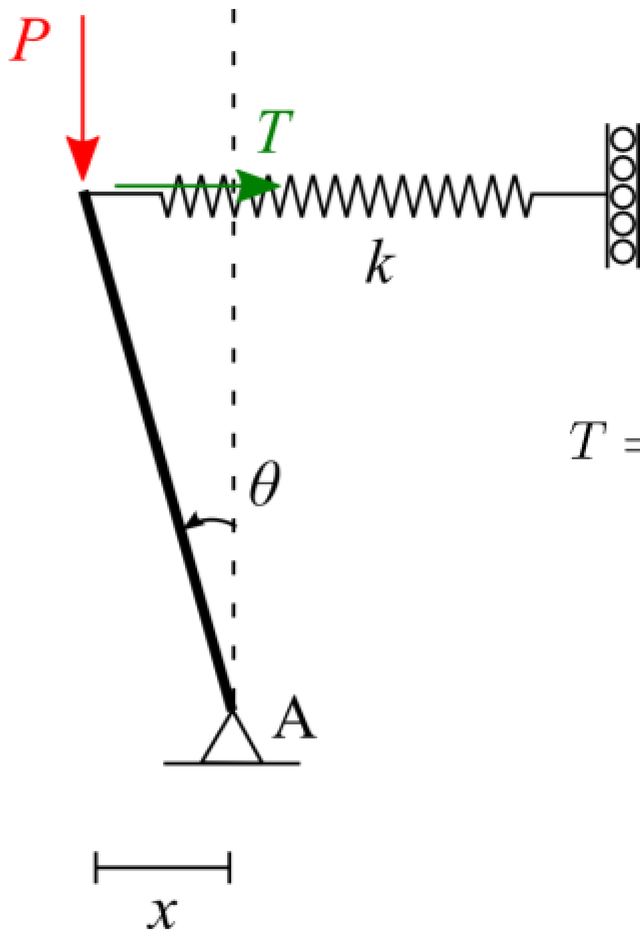
$$\left(\sum_i \boldsymbol{F}^{(i)}\right) \cdot \widetilde{\boldsymbol{v}}^O + \left(\sum_i \boldsymbol{r}^{(O,i)} \times \boldsymbol{F}^{(i)}\right) \cdot \widetilde{\boldsymbol{\omega}}^O = 0, \forall \widetilde{\boldsymbol{v}}^O, \widetilde{\boldsymbol{\omega}}^O$$

Rigid body: $\widetilde{\boldsymbol{v}}^{(i)} = \widetilde{\boldsymbol{v}}^O + \widetilde{\boldsymbol{\omega}}^O \times \boldsymbol{r}^{(O,i)}$

$$\sum_i \boldsymbol{F}^{(i)} \cdot \widetilde{\boldsymbol{v}}^{(i)} = 0, \forall \widetilde{\boldsymbol{v}}^{(i)} \iff \boxed{\sum_i \widetilde{\mathcal{P}}^{(i)} = 0, \forall \widetilde{\boldsymbol{v}}^{(i)}}$$

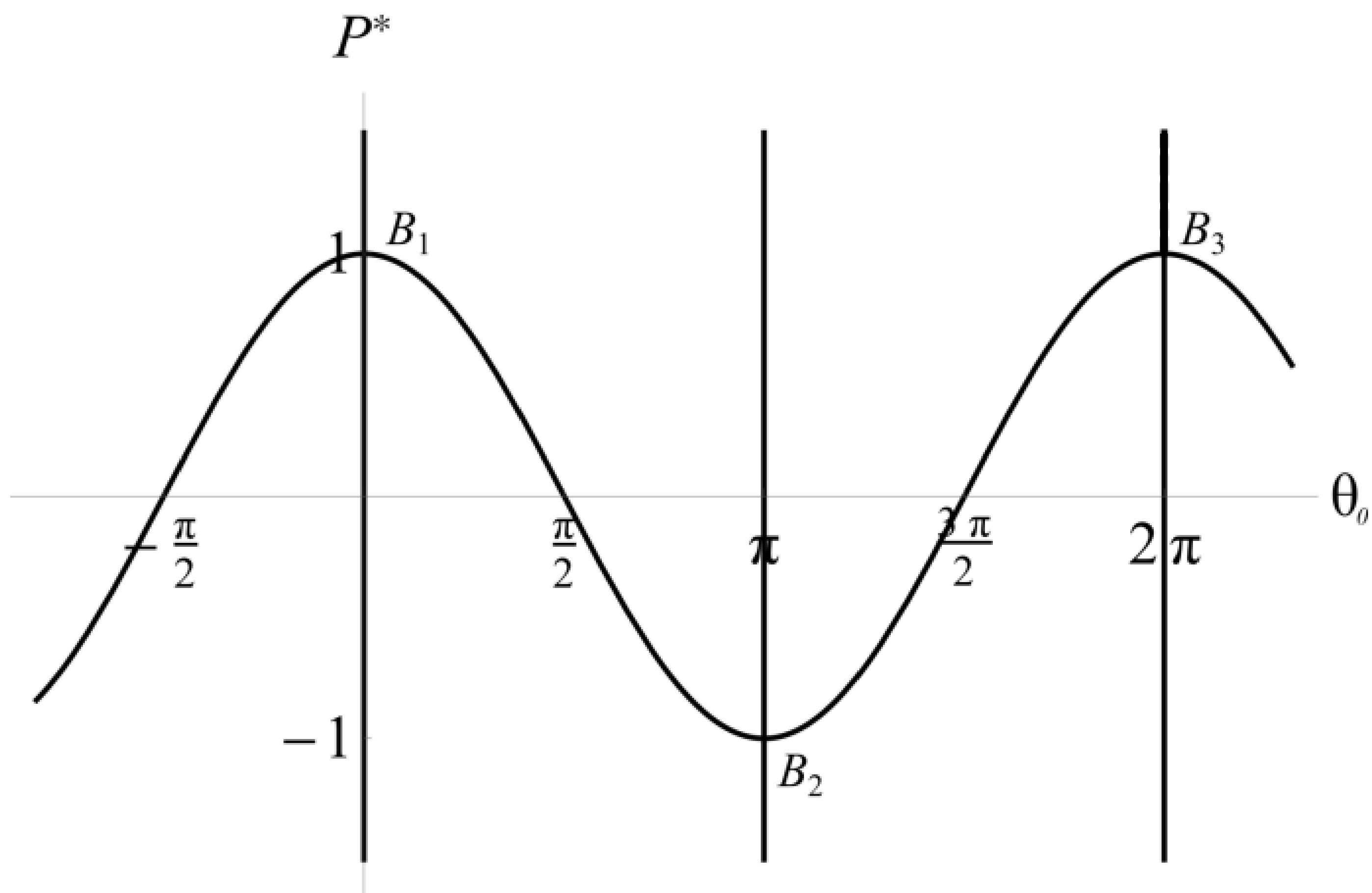
# Exercise #1

*-> Using the PPV find all the equilibrium points (angles ϑ) of the system*:

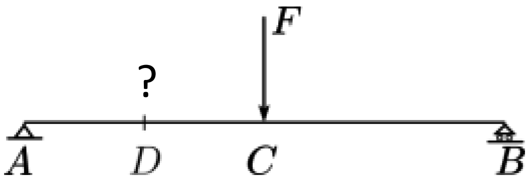


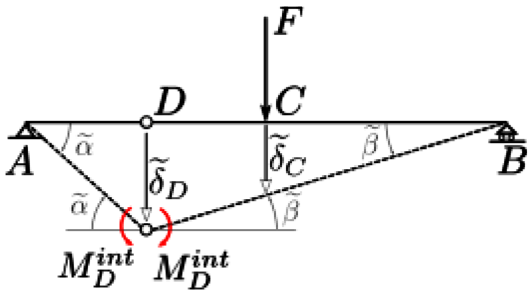

$T = kx$

# Answer #1



# PVP applied to a system of bodies




$$\sum_i \widetilde{\mathcal{P}}^{(i)} = \sum_j \widetilde{\mathcal{P}}^{(ext,j)} - \sum_k \widetilde{\mathcal{P}}^{(int,k)} = 0, \qquad \forall \widetilde{\boldsymbol{v}}^{(i)}$$

$-$

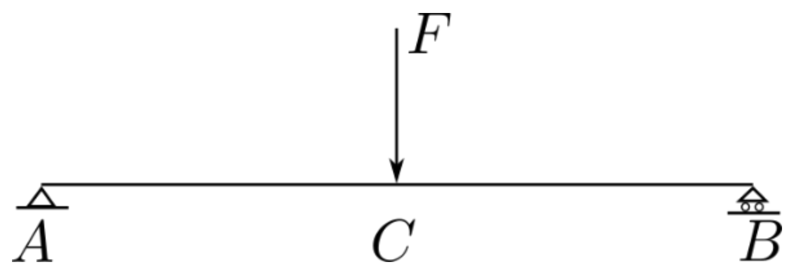
$$\widetilde{\delta}_D = \widetilde{\alpha}\frac{L}{4} = \widetilde{\beta}\frac{3L}{4}$$

$$\widetilde{\delta}_C = \widetilde{\beta}\frac{L}{2}$$

$$\overset{\forall \widetilde{\beta}}{\Longrightarrow} \quad M_D^{int} = \frac{FL}{8}$$

# Exercise #2

-> *Using the PPV find the internal moment at point C:*



# Answer #2

$$M_D^{int} = \frac{FL}{4}$$

**INTERNAL ENERGY NOT TO FORGET IS**

$$\widetilde{\mathcal{P}}^{(i)} = \widetilde{\mathcal{P}}^{(ext,j)} - \widetilde{\mathcal{P}}^{(int,k)} = 0$$

$$(F_B \widetilde{u}_B + F_G \widetilde{u}_A) - \sigma\widetilde{\varepsilon} = 0$$


**BAD**

**GOOD**

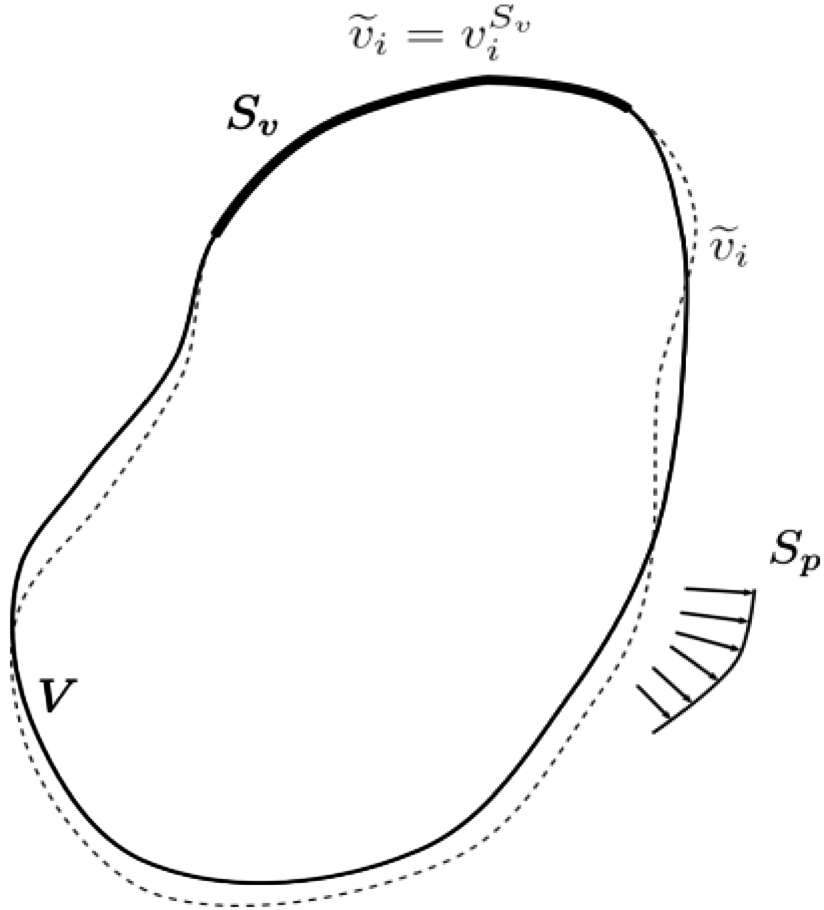
$F_B$

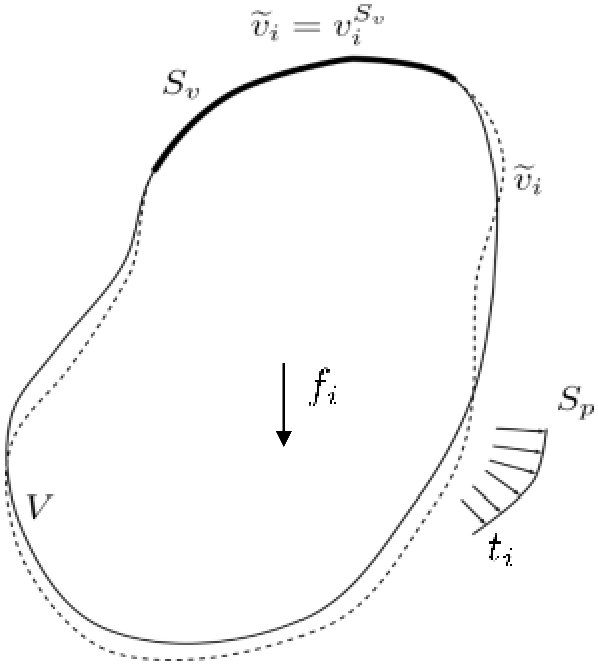
$F_G$


KIDS!!!
STAR WARS

# Continuum Mechanics

# PVP and continuum mechanics






# Definitions

internal deformation enery density: $\widetilde{p}^{(int)} = \sigma_{ij}\widetilde{\varepsilon}_{ij}$

$$\widetilde{\varepsilon}_{ij} = \frac{1}{2}\left(\widetilde{v}_{i,j} + \widetilde{v}_{j,i}\right), \quad \sigma_{ij} = \sigma_{ji}$$

external enery density: $\widetilde{p}^{(ext,t)} = t_i\widetilde{v}_i$

$$\widetilde{p}^{(ext,f)} = f_i\widetilde{v}_i$$

$$\int_V \sigma_{ij}\widetilde{\varepsilon}_{ij}dV - \int_V f_i\widetilde{v}_i dV - \int_{\partial V} t_i\widetilde{v}_i dS = 0, \forall\widetilde{v}_i$$

$$\int_V \sigma_{ij}\widetilde{\varepsilon}_{ij}dV = \int_V \sigma_{ij}\widetilde{v}_{i,j}dV = \int_V (\sigma_{ij}\widetilde{v}_i)_{,j}dV - \int_V \sigma_{ij,j}\widetilde{v}_i dV$$

Using the divergence theorem:

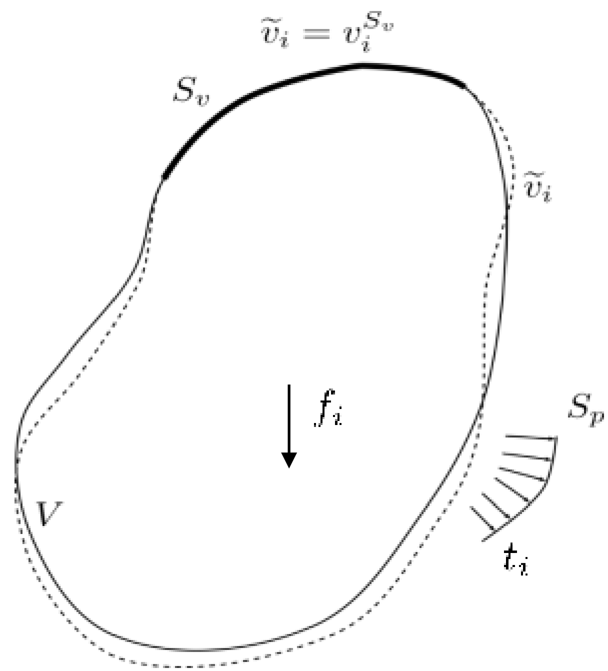
$$\int_V (\sigma_{ij}\widetilde{v}_i)_{,j}dV = \int_{\partial V} (\sigma_{ij}\widetilde{v}_i)n_j dS = \int_{\partial V} (\sigma_{ij}n_j)\widetilde{v}_i dS$$

Replacing:

$$\int_V (\sigma_{ij,j} + f_i)\,\widetilde{v}_i dV + \int_{\partial V} (t_i - \sigma_{ij}n_j)\,\widetilde{v}_i dS = 0, \forall\widetilde{v}_i$$

$$\int_V (\sigma_{ij,j} + f_i)\, \widetilde{v}_i dV + \int_{\partial V} (t_i - \sigma_{ij} n_j)\, \widetilde{v}_i dS = 0, \forall \widetilde{v}_i$$

$$\overset{\forall V}{\Longleftrightarrow} \begin{cases} \sigma_{ij,j} + f_i = 0 \\ t_i = \sigma_{ij} n_j \end{cases}$$



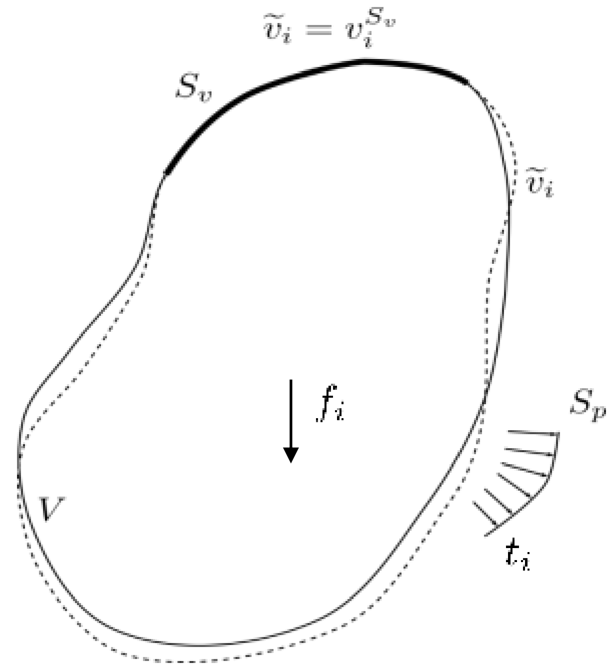

Which are the classical equilibrium equations in continuum mechanics
-> Strong form

# Linear & angular momentum balance principles

- $\int_{\partial V} t_i dS + \int_V f_i dV = 0$

with $t_i = \sigma_{ij} n_j$

- $\int_{\partial V} \epsilon_{ijk} t_j x_k dS + \int_{\partial V} \epsilon_{ijk} f_j x_k dV = 0$




$$\int_{\partial V} \boldsymbol{t} dS + \int_V \boldsymbol{f} dV = \int_{\partial V} \vec{t} dS + \int_V \vec{f} dV$$

$$\int_{\partial V} \boldsymbol{t} \times \boldsymbol{x} dS = \int_{\partial V} \vec{t} \times \vec{x} dS$$

$$\int_{\partial V} t_i dS + \int_V f_i dV = 0$$

$$t_i = \sigma_{ij} n_j$$

Using the divergence theorem:

$$\int_{\partial V} t_i dS = \int_{\partial V} \sigma_{ij} n_j dS = \int_V \sigma_{ij,j} dV$$

Replacing:

$$\int_V (\sigma_{ij,j} + f_i) dV = 0$$

$$\int_{\partial V} \epsilon_{ijk} t_j x_k dS + \int_V \epsilon_{ijk} f_j x_k dV = 0$$

$$t_i = \sigma_{ij} n_j$$

Using the divergence theorem:

$$\int_{\partial V} \epsilon_{ijk} t_j x_k dS = \int_{\partial V} \epsilon_{ijk} \sigma_{jp} x_k n_p dS = \int_V (\epsilon_{ijk} \sigma_{jp} x_k)_{,p} dV =$$

$$= \int_V (\epsilon_{ijk} \sigma_{jp} x_k)_{,p} dV = \int_V \epsilon_{ijk} \sigma_{jp,p} x_k dV + \int_V \epsilon_{ijk} \sigma_{jp} x_{k,p} dV =$$

$$= \int_V \epsilon_{ijk} (-f_j) x_k dV + \int_V \epsilon_{ijk} \sigma_{jp} \delta_{kp} dV$$

$$= -\int_V \epsilon_{ijk} f_j x_k dV + \int_V \epsilon_{ijk} \sigma_{jk} dV$$

Replacing:

$$\int_{\partial V} \epsilon_{ijk} t_j x_k dS + \int_V \epsilon_{ijk} f_j x_k dV = \int_V \epsilon_{ijk} \sigma_{jk} dV = 0$$

$$\int_V \epsilon_{ijk} \sigma_{jk} dV = 0$$

$$\boxed{\sigma_{jk} = \sigma_{kj}}$$

## PVP

Starting point:

- Virtual (generalized) displacements
- Virtual powers (external & internal)

Less intuition

Prone to the Galerkin Method (Finite Elements)

## Momentum Balance

Starting point:

- (Generalized) stresses

More intuition

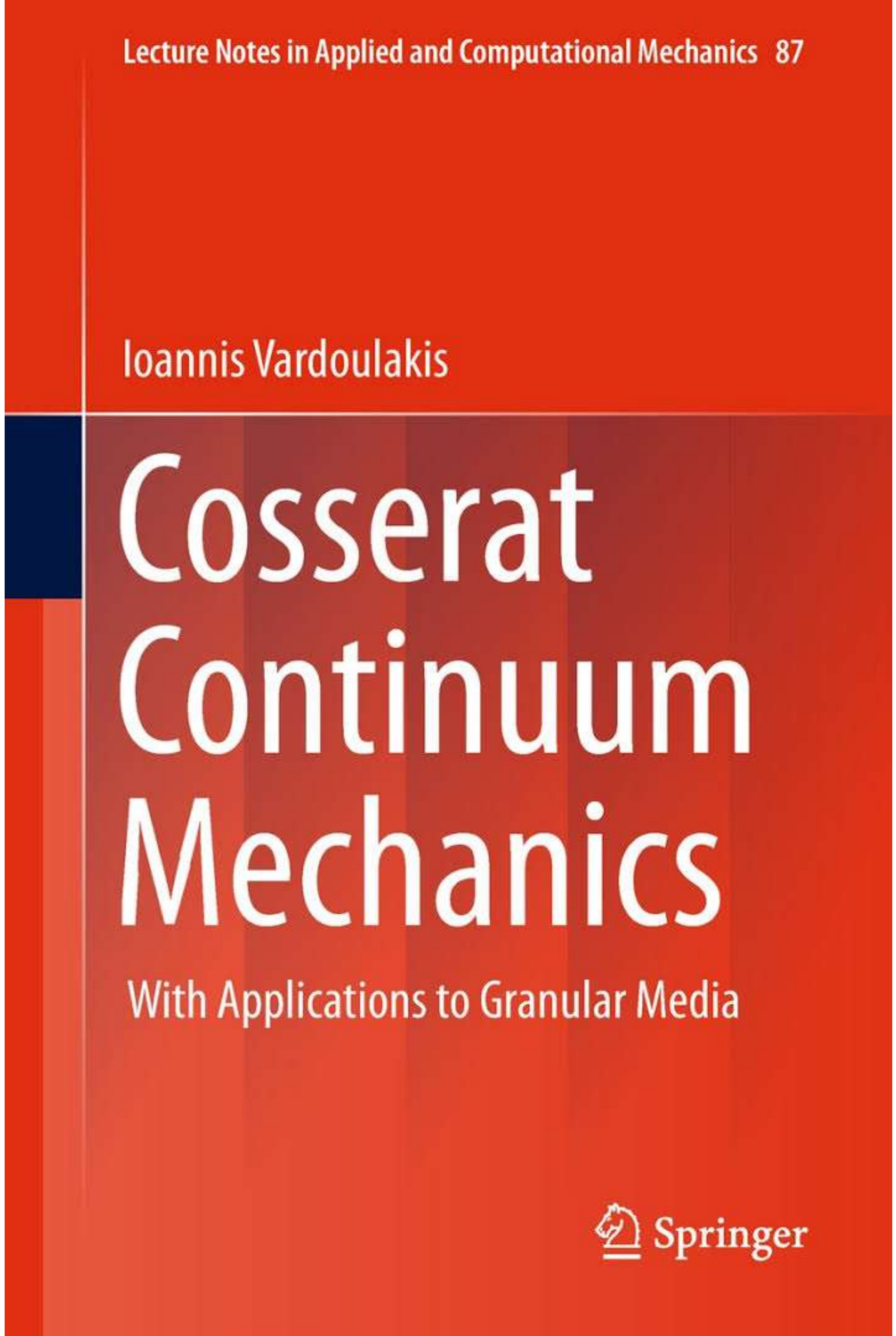
**Principle of Virtual Powers**

**or**

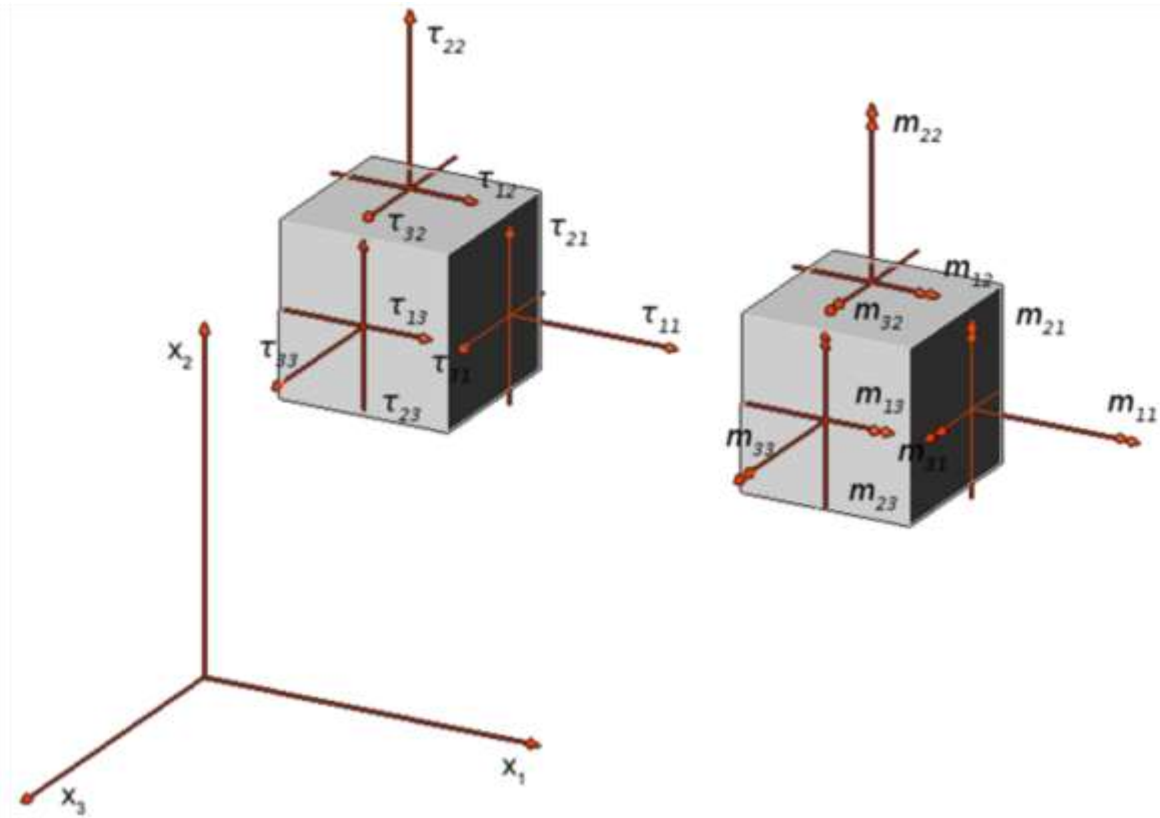
**Principle linear & angular momentum balance ?**

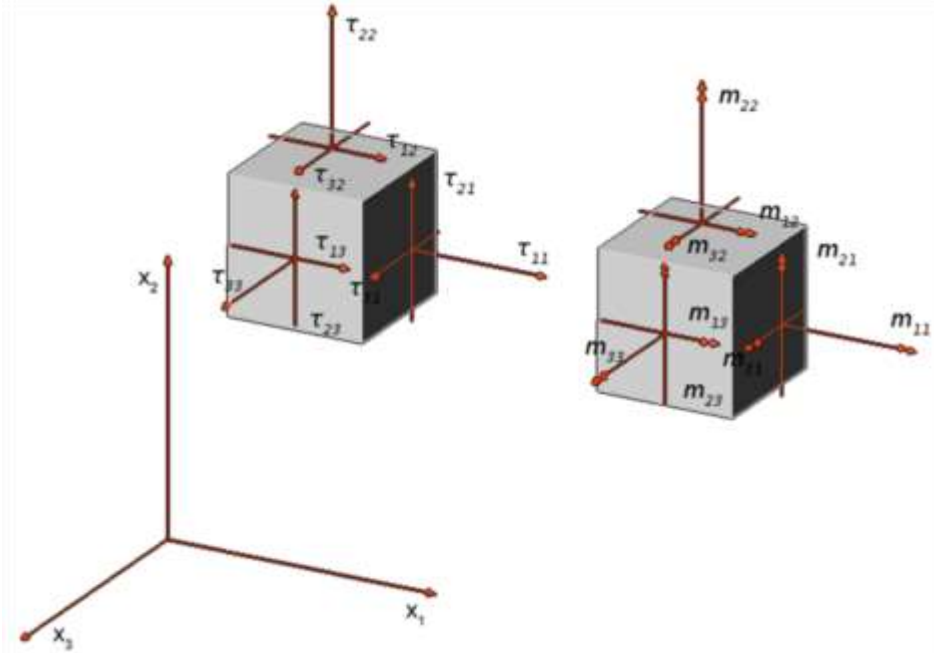
**It is a matter of principles!**

# Cosserat Continuum


Lecture Notes in Applied and Computational Mechanics 87
Ioannis Vardoulakis
Cosserat Continuum Mechanics
With Applications to Granular Media
Springer

# From the MB point of view



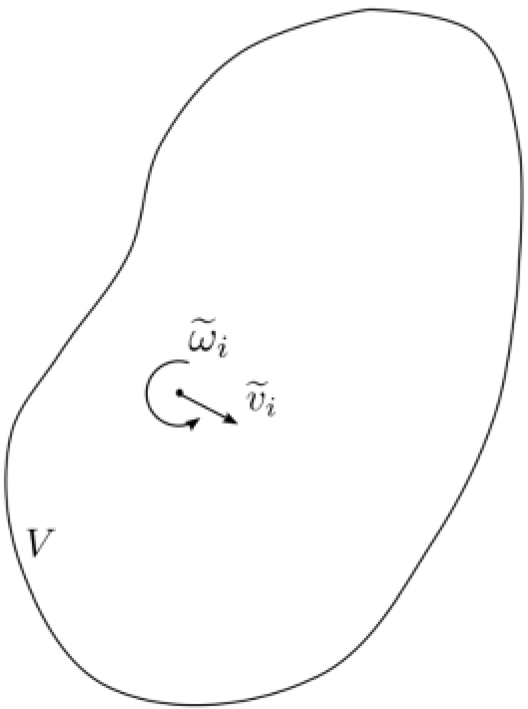



Equilibrium on infinitesimal volume
& Cauchy tetrahedron:

$$\tau_{ij,j} + f_i = 0, \qquad t_i = \tau_{ij} n_j$$

$$m_{ij,j} - \epsilon_{ijk}\tau_{jk} + \psi_i = 0, \qquad \mu_i = m_{ij} n_j$$

# Energetic approach: PVP




$$\widetilde{p}^{(ext,f)} = f_i \widetilde{v}_i + \psi_i \widetilde{\omega}_i^c$$

$$\widetilde{p}^{(ext,t)} = t_i \widetilde{v}_i + \mu_i \widetilde{\omega}_i^c$$

$$\widetilde{p}^{int} = \tau_{ij} \widetilde{\gamma}_{ij} + m_{ij} \widetilde{k}_{ij}$$

$$\gamma_{ij} = u_{i,j} + \epsilon_{ijk} \omega_k^c$$

$$k_{ij} = \omega_{i,j}^c$$

# Exercise #3

*-> Retrieve the strong form of equilibrium equations for Cosserat:*

$$\widetilde{p}^{(ext,f)} = f_i\widetilde{v}_i + \psi_i\widetilde{\omega}_i^c$$

$$\widetilde{p}^{(ext,t)} = t_i\widetilde{v}_i + \mu_i\widetilde{\omega}_i^c$$

$$\widetilde{p}^{int} = \tau_{ij}\widetilde{\gamma}_{ij} + m_{ij}\widetilde{k}_{ij}$$

$$\gamma_{ij} = u_{i,j} + \epsilon_{ijk}\omega_k^c$$

$$k_{ij} = \omega_{i,j}^c$$

$$\Rightarrow$$

$$\tau_{ij,j} + f_i = 0$$

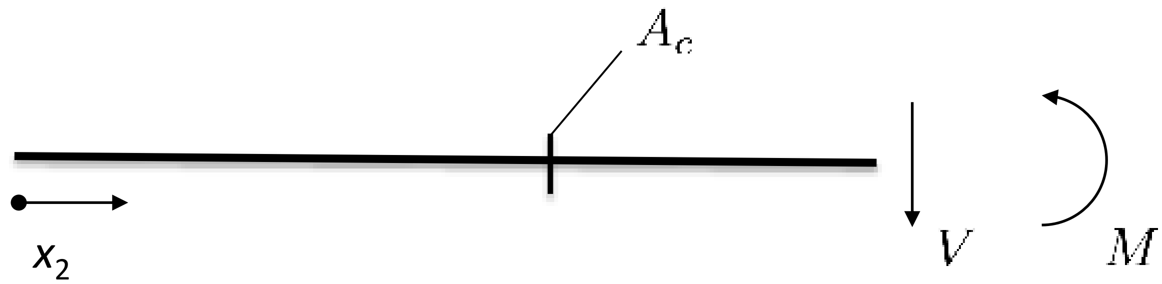
$$m_{ij,j} - \epsilon_{ijk}\tau_{jk} + \psi_i = 0$$

$$t_i = \tau_{ij}n_j$$

$$\mu_i = m_{ij}n_j$$

# 1D Cosserat: The Timoshenko beam

Consider a narrow infinite strip of Cosserat continuum in the $x_2$-direction, invariant in the $x_1$- and $x_3$-directions:

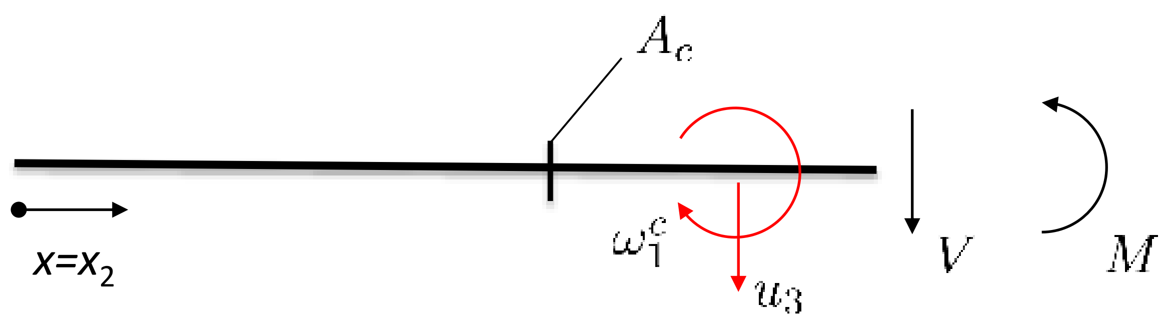



$$\left.\begin{aligned} \tau_{ij,j} &= 0 \\ m_{ij,j} - \epsilon_{ijk}\tau_{jk} &= 0 \end{aligned}\right\} \qquad \begin{aligned} \tau_{32,2} &= 0 \\ m_{12,2} + \tau_{32} &= 0 \end{aligned}$$

Call: $\begin{aligned} V &\equiv \tau_{32} A_c \\ M &\equiv -m_{12} A_c \\ x_2 &\equiv x \end{aligned} \longrightarrow \begin{aligned} \frac{dV}{dx} &= 0 \\ \frac{dM}{dx} &= V \end{aligned}$

$$\frac{dV}{dx} = 0$$

$$\frac{dM}{dx} = V$$




Kinematics:

$$\gamma_{32} = u_{3,2} - \omega_1^c$$

$$k_{12} = \omega_{1,2}^c \equiv \kappa \quad \text{(curvature)}$$

Constitutive law:

$$\tau_{32} = G\gamma_{32}$$

$$m_{12} = Ck_{12}$$

$$\longrightarrow \begin{cases} V = G^\star \frac{du_3}{dx}, & G^\star \equiv GA_c \\ M = -C^\star \kappa, & C^\star \equiv CA_c \equiv EI \end{cases}$$

# When to use Cosserat?

- Size of the microstructure is important
  (compared to the characteristic wave length of the loading)

- Presence of internal lengths

- There is no scale separation

- Important stress gradients
  (compared to the size of the microstructure)

-> Shear bands, high-frequency phenomena, THMC couplings

# Upscaling to Cosserat continuum?

The target is to derive the CC constitutive law based on the properties of the microstructure.

Not trivial task as **classical asymptotic homogenization approaches are hardly applicable** (no scale separation).

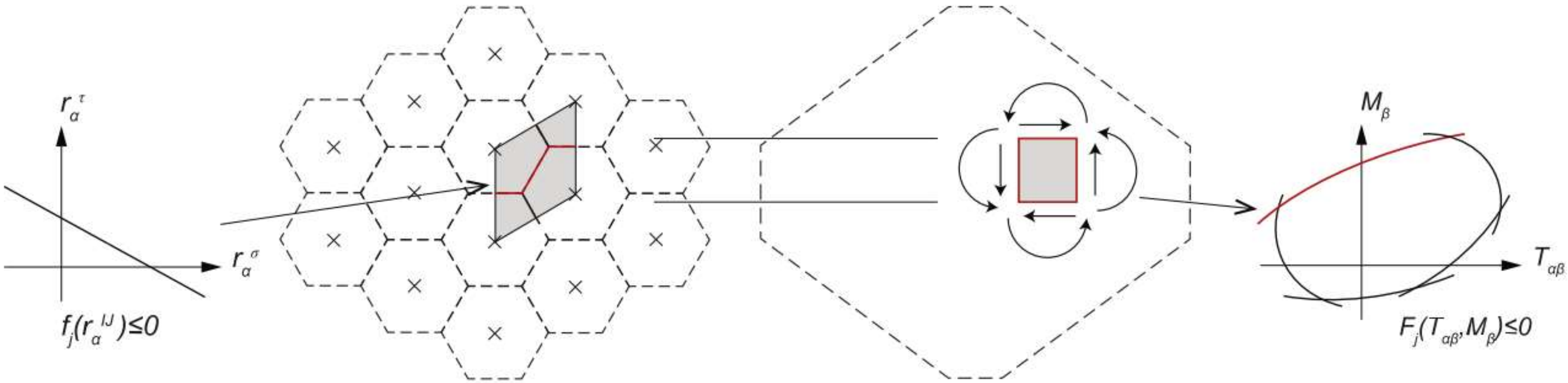
Equivalent continuum approach:

Construct a continuum that
1) Has the **same energy** with the discrete
2) **Represents the kinematics** of the discrete

**PVP** → Relation between generalized stresses and generalized deformations i.e. **Constitutive law**

[Forest et al. 1998, Bardet & Vardoulakis, 2001, Chang & Kung 2005, Gerolymatou, 2011, Godio et al., 2015,16, …]

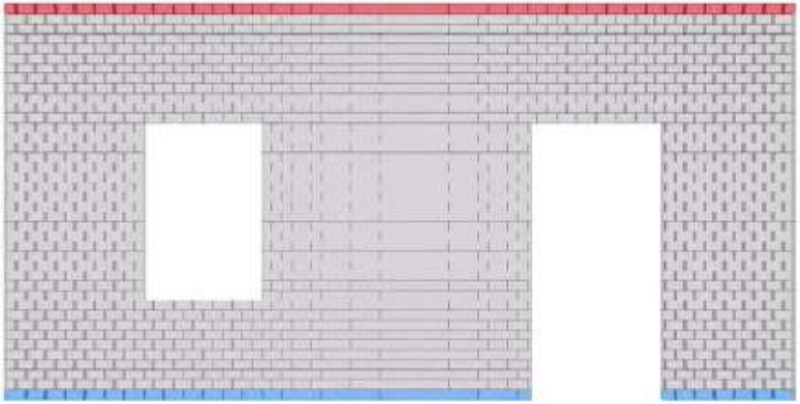
# How can we describe multiple failure mechanisms with Cosserat continua?



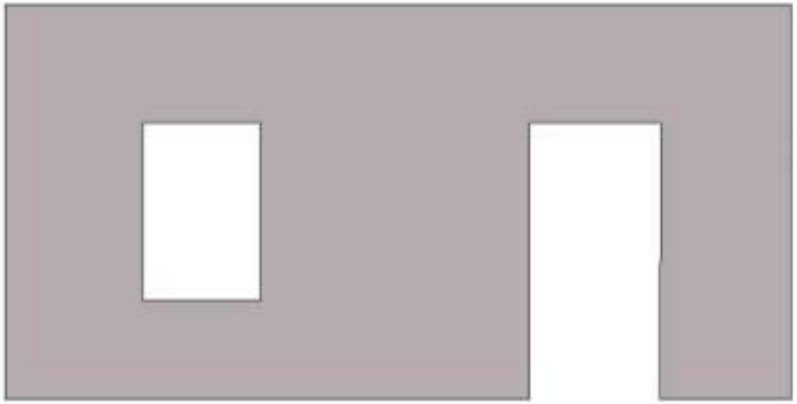

Upscaling procedure and transfer of internal lengths

[Forest et al., 1998, 1999, 2001]
[Stefanou et al.,2008, Godio et al. 2014, 2015, 2016a,b, 2017]

# Cosserat for masonry



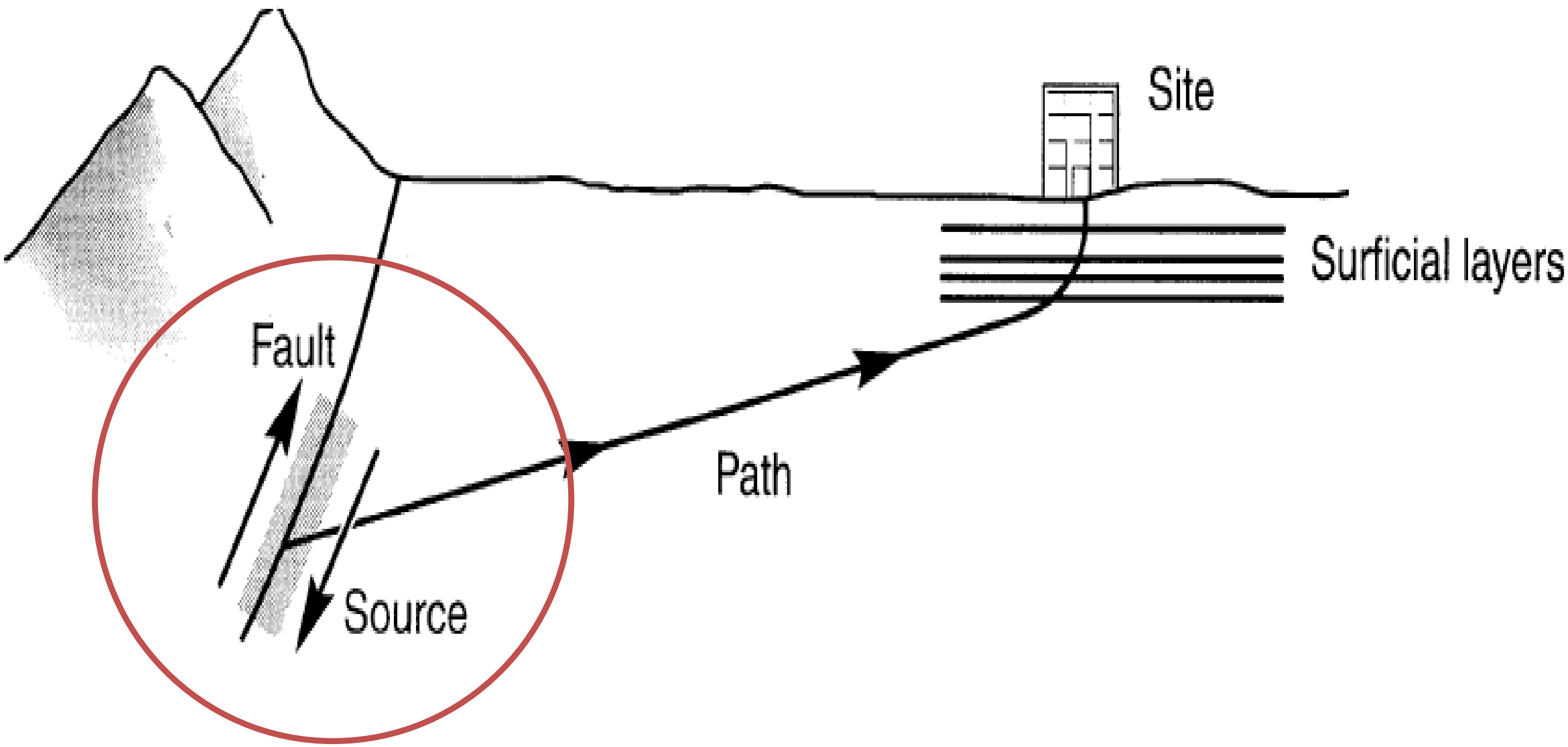
*discrete elements model*



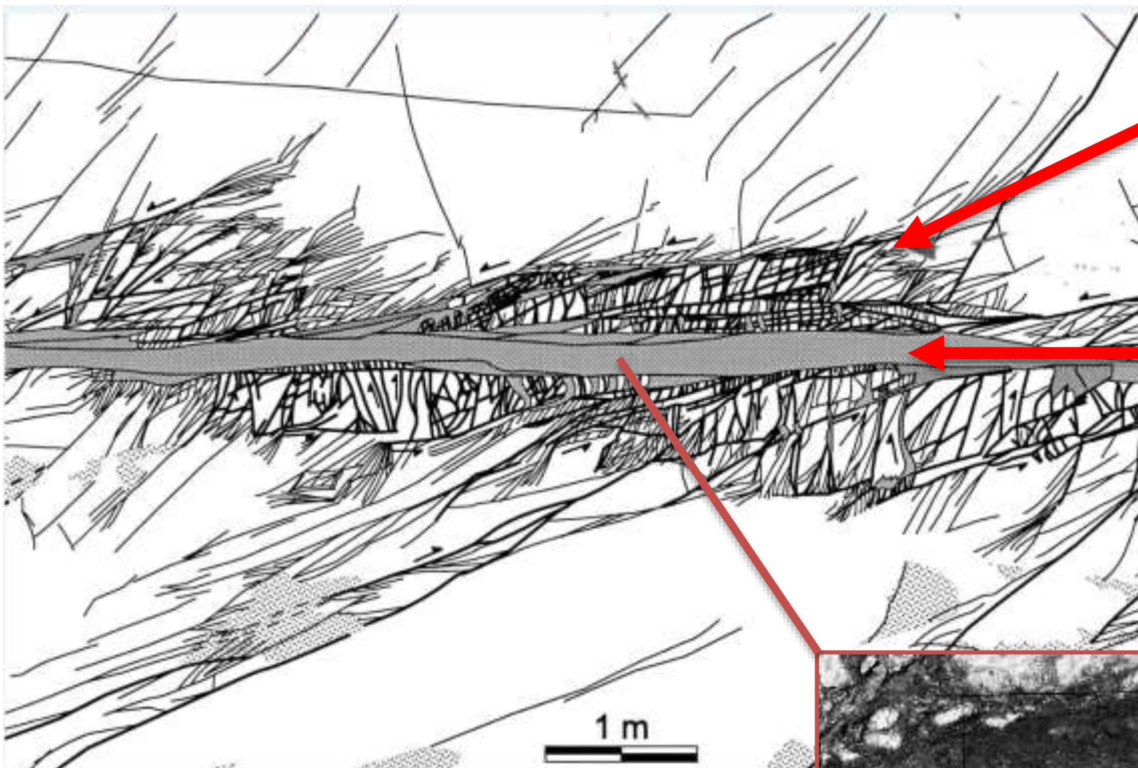
*Cosserat model*

Masonry is a great toy model for developing higher order theories

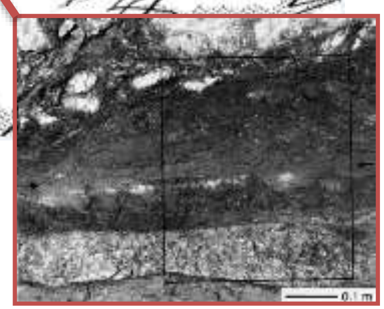
# Cosserat for EQ faults




[courtesy: Kramer 1996]




[Myers et al., 1994]



[Chester & Chester, 1998]

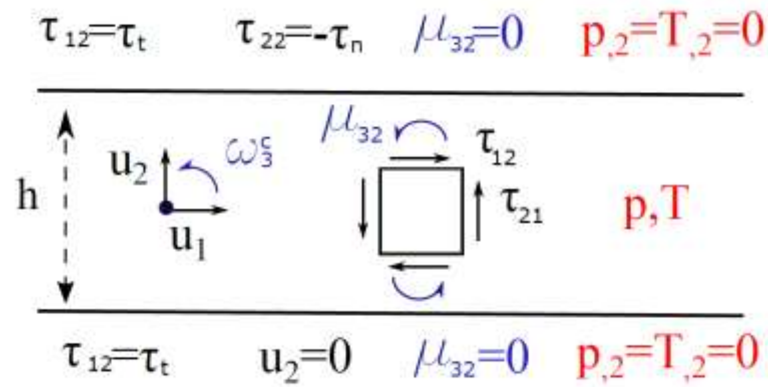
**Damaged zone**

thickness: from ~10 m to ~1 km

**Gouge**

composed of very fine crushed particles, where the slip is localized thickness: from ~1 μm to ~10 mm

A fault zone is modelled as an infinite layer under shear:




Momentum balance equations:

$$\tau_{ij,j} - \rho \frac{\partial^2 U_i}{\partial t^2} = 0$$

$$\mu_{ij,j} - e_{ijk}\tau_{jk} - \rho I \frac{\partial^2 \omega_i^c}{\partial t^2} = 0$$

Elasto-plastic constitutive equation:

$$\dot{\tau}'_{ij} = C^{ep}_{ijkl}\ \dot{\gamma}_{kl} + D^{ep}_{ijkl}\ \dot{\kappa}_{kl} + E^{ep}_{ijkl}\ \dot{T}\ \delta_{kl}$$

$$\dot{\mu}_{ij} = M^{ep}_{ijkl}\ \dot{\kappa}_{kl} + L^{ep}_{ijkl}\ \dot{\gamma}_{kl} + N^{ep}_{ijkl}\ \dot{T}\ \delta_{kl}$$

Terzaghi effective stress: $\tau'_{ij} = \tau_{ij} + p\ \delta_{ij}$

Energy balance equation:

$$\frac{\partial T}{\partial t} - c_{th} T_{,ii} = \frac{1}{\rho C}\underbrace{(\sigma_{ij}\dot{\varepsilon}^p_{ij} + \mu_{ij}\dot{\kappa}^p_{ij})}_{\text{Plastic work}}$$

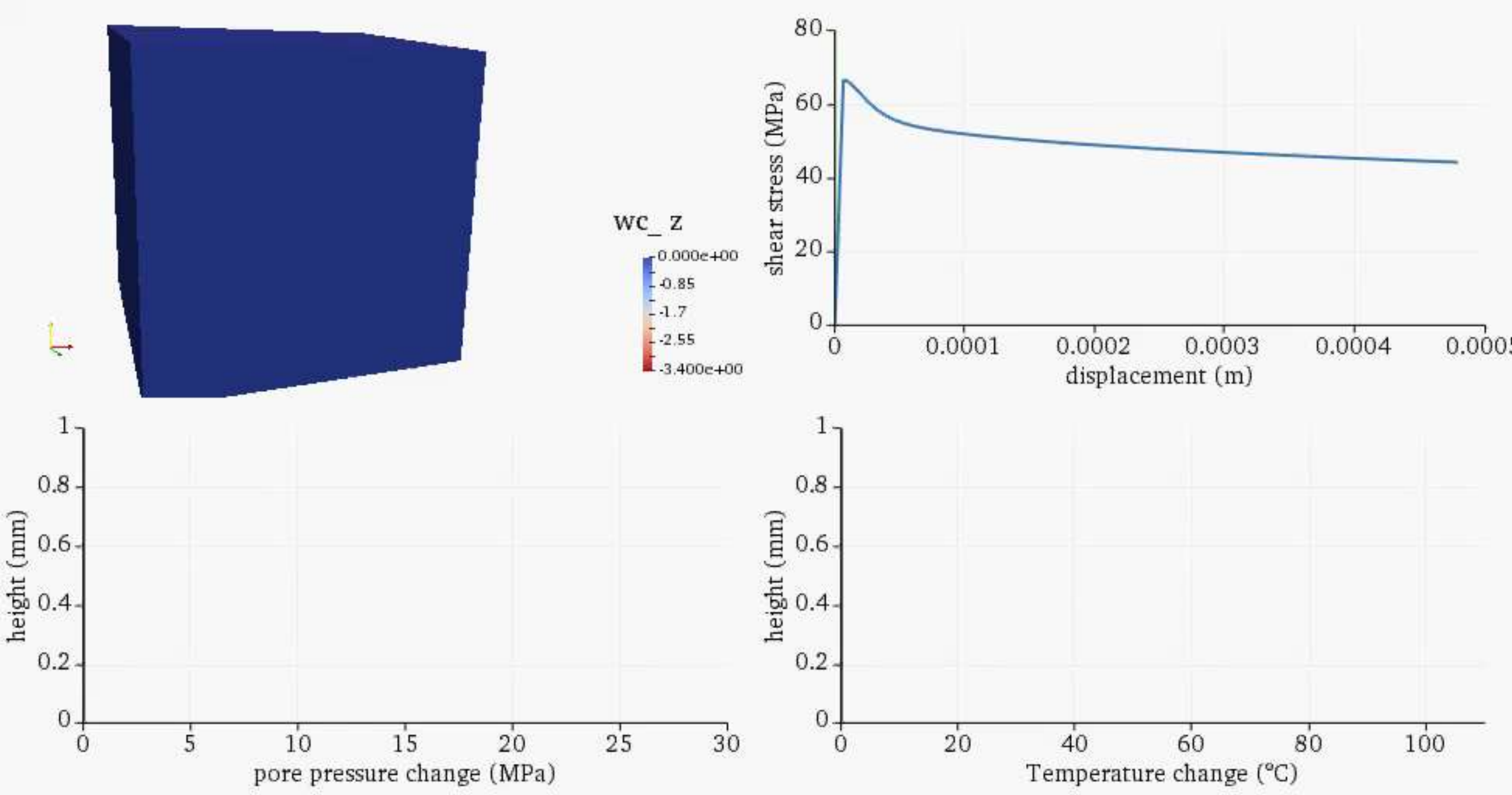
Mass balance equation:

$$\frac{\partial p}{\partial t} = c_{hy}\ p_{,ii} + \underbrace{\frac{\lambda^*}{\beta^*}\frac{\partial T}{\partial t}}_{\text{Thermal pressurisation}} - \underbrace{\frac{1}{\beta^*}\frac{\partial \varepsilon_v}{\partial t}}_{\text{Porosity variation}}$$

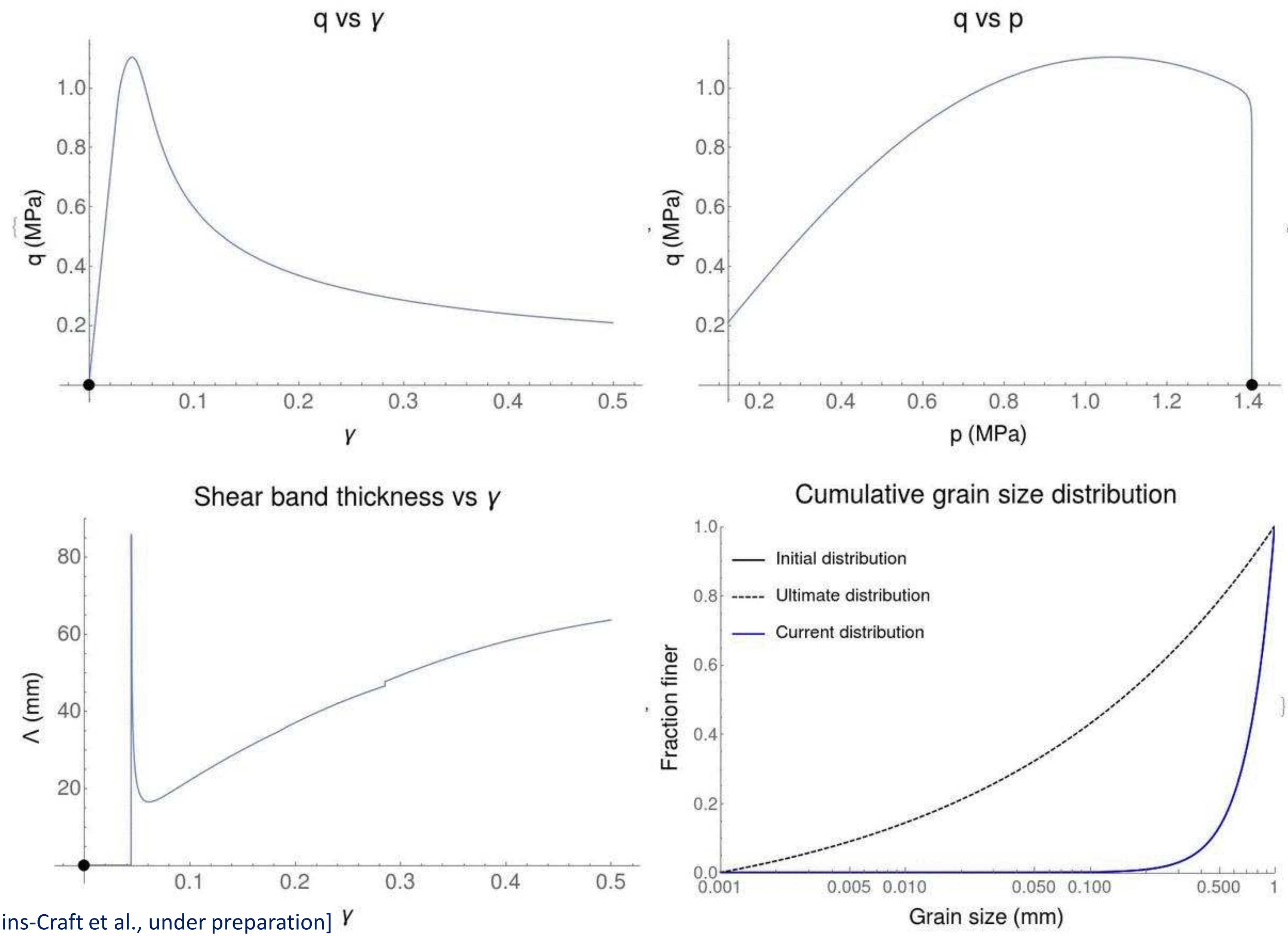
[Rattez et al. 2018a,b,c]

# Numerical results

Fast rate (1 m/s)



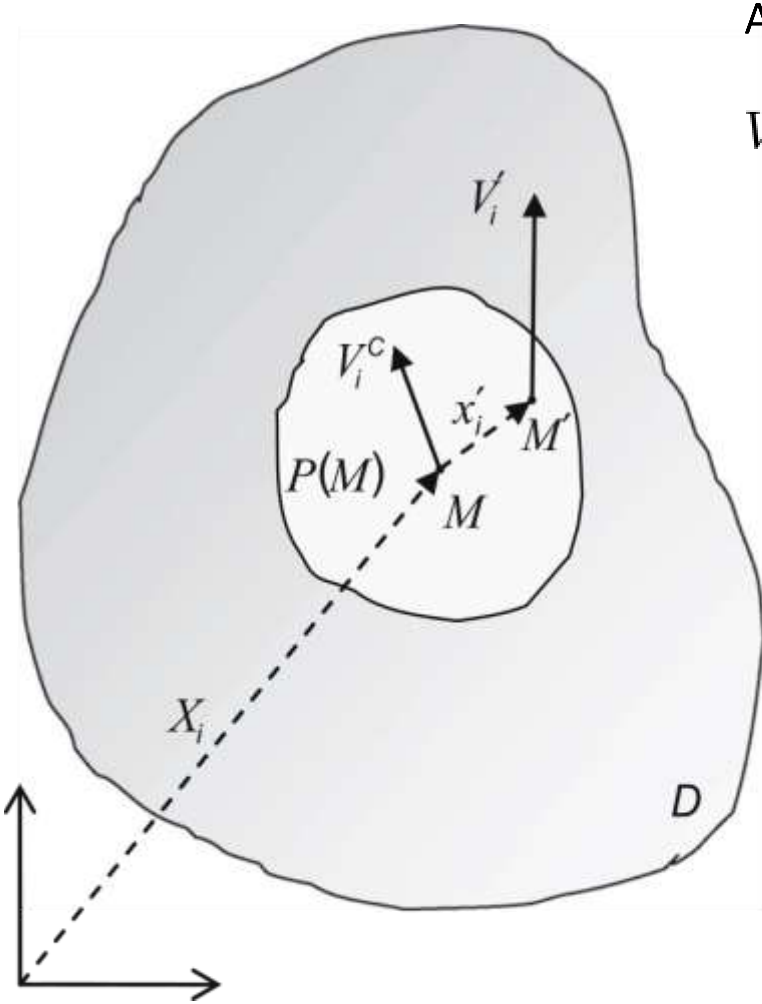
# Grain size evolution based on Breakage Mechanics




[Collins-Craft et al., under preparation]

# Micromorphic (generalized) Continua

# The starting hypotheses




Ansatz:

$$V'_i = V_i + \chi_{ij}x'_j + \chi_{ijk}x'_jx'_k + \chi_{ijkl}x'_jx'_kx'_l + \ldots$$

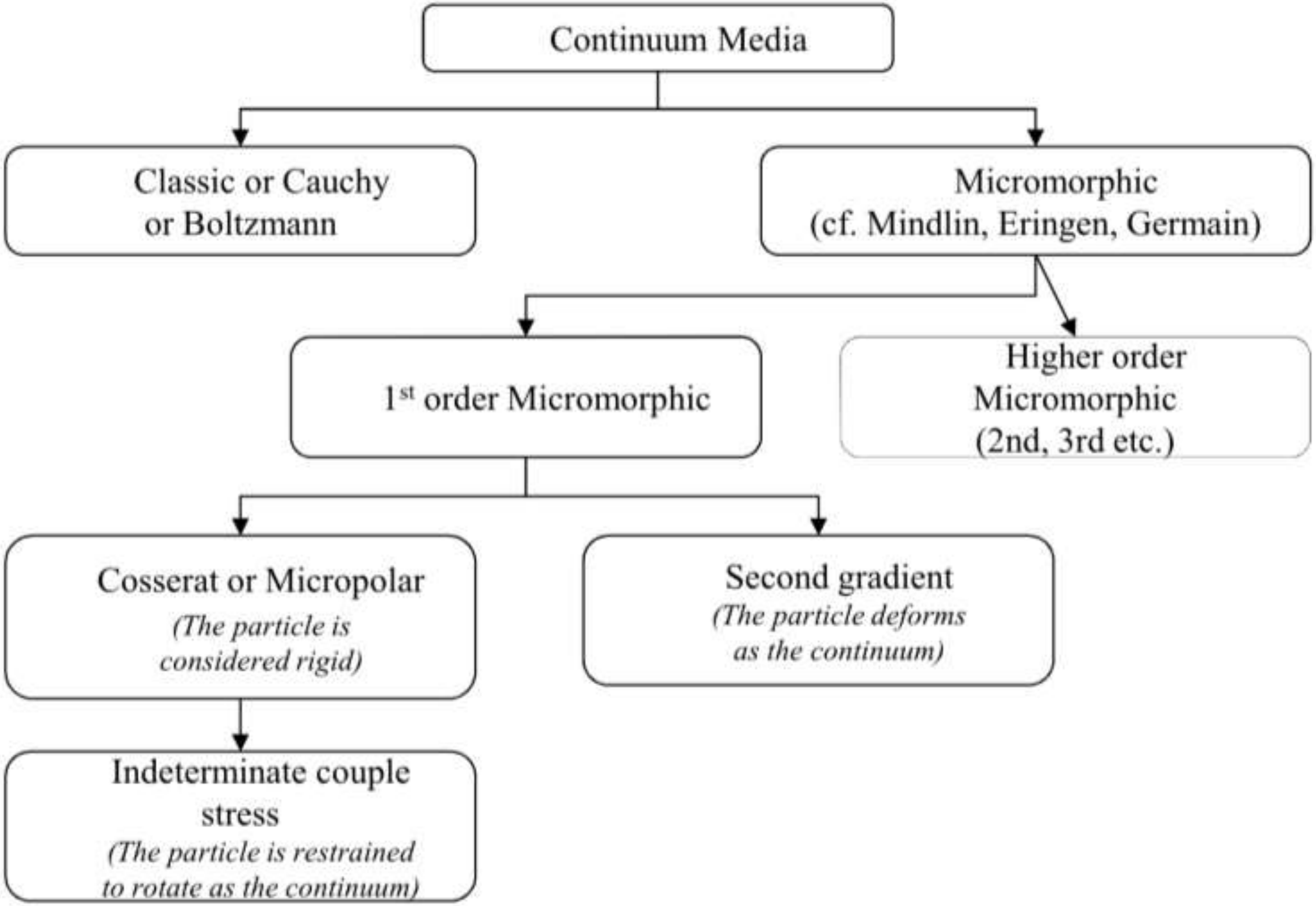
$$\widetilde{p}^{(ext,t)} = t_i\widetilde{v}_i + \mu_{ij}\widetilde{\chi}_{ij} + \mu_{ijk}\widetilde{\chi}_{ijk} + \ldots$$

$$\widetilde{p}^{(ext,f)} = f_i\widetilde{v}_i + \psi_{ij}\widetilde{\chi}_{ij} + \psi_{ijk}\widetilde{\chi}_{ijk} + \ldots$$

$$\begin{aligned}\widetilde{p}^{int} &= \tau_{ij}\widetilde{V}_{i,j} \\ &- (s_{ij}\widetilde{\chi}_{ij} + s_{ijk}\widetilde{\chi}_{ijk} + \ldots) \\ &+ (\nu_{ijk}\widetilde{\kappa}_{ijk} + \nu_{ijkl}\widetilde{\kappa}_{ijkl} \ldots)\end{aligned}$$

[Germain, 1973, Mindlin, 1964 Eringen, 1999, …]

# Classification



# Applying PVP: strong form…

$$
\begin{aligned}
\tau_{ij,j} + f_i = 0, &\qquad t_i = \tau_{ij} n_j \\
\nu_{ijk,k} + s_{ij} + \psi_{ij} = 0, &\qquad \mu_{ij} = \nu_{ijk} n_k \\
\nu_{ijkl,l} + s_{ijk} + \psi_{ijk} = 0, &\qquad \mu_{ijk} = \nu_{ijkl} n_l \\
\ldots &
\end{aligned}
$$

# Exercise #4

-> *From the general micromorphic equations and by setting:*

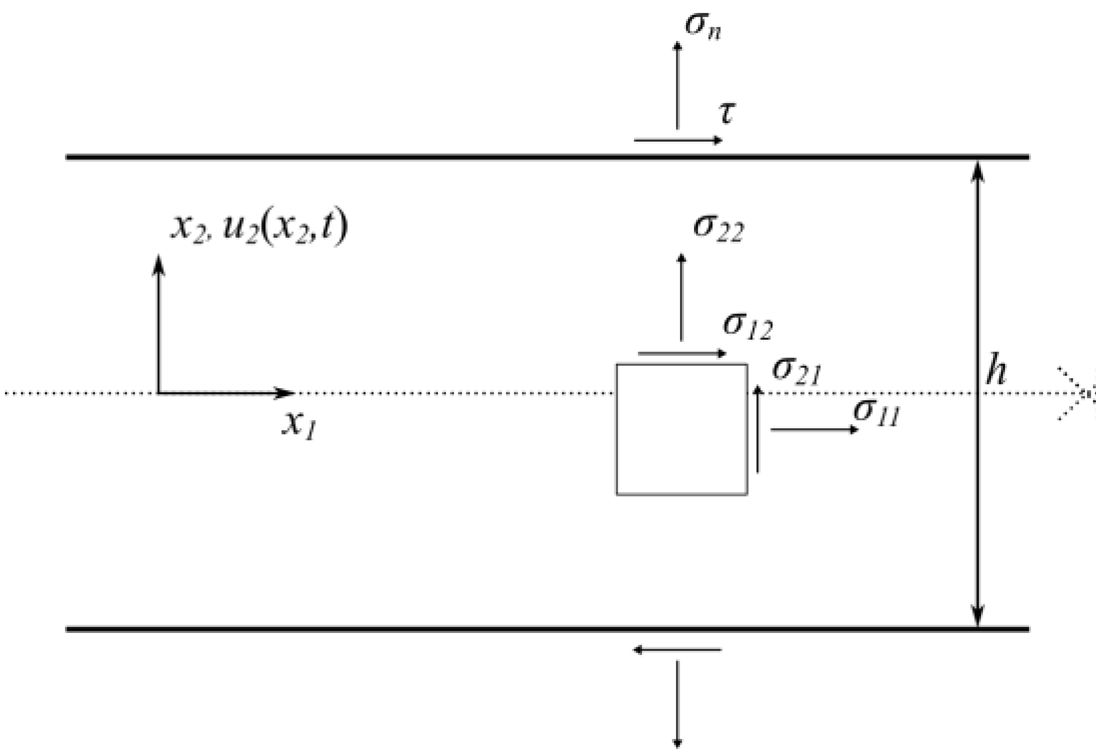
$$\chi_{ij} = -\epsilon_{ijk}\omega^c_k,\ k_{ij} = \omega^c_{i,j},\ s_{ij} = -\frac{1}{2}\epsilon_{ijk}s_k,\ \mu_{ij} = -\frac{1}{2}\epsilon_{ijk}\mu_k,$$
$$\nu_{ijk} = -\frac{1}{2}\epsilon_{ijl}m_{lk},\ \psi_{ij} = -\frac{1}{2}\epsilon_{ijk}\psi_k,\ \tau_{ij} \equiv \sigma_{ij} + s_{ij}$$

*retrieve the strong form of equilibrium equations for Cosserat.*

(hint: $\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$ )

# Finite Elements

# Example: *Simple shear with Cauchy continuum*




*Virtual power densities:*

$$\widetilde{p}^{(int)} = \sigma_{22}\widetilde{v}_{2,2} + \sigma_{12}\widetilde{v}_{1,2}$$
$$\widetilde{p}^{(ext)} = \sigma_n\widetilde{v}_2 + \tau\widetilde{v}_1$$

*Constitutive law:*

$$\sigma_{22} = M_{el}u_{2,2}$$
$$\sigma_{12} = G_{el}u_{1,2}$$

*PVP:*

$$\widetilde{\mathcal{P}} = 0, \forall\widetilde{v}_i \iff \int_D \widetilde{p}^{(int)}dV = \int_{\partial D} \widetilde{p}^{(ext)}dS, \forall\widetilde{v}_i$$

# Example: *Simple shear with Cauchy continuum*

$$\widetilde{p}^{(int)} = \sigma_{22}\widetilde{v}_{2,2} + \sigma_{12}\widetilde{v}_{1,2} \qquad \sigma_{22} = M_{el}u_{2,2}$$

$$\widetilde{p}^{(ext)} = \sigma_n\widetilde{v}_2 + \tau\widetilde{v}_1 \qquad \sigma_{12} = G_{el}u_{1,2}$$

$$\widetilde{\mathcal{P}} = 0, \forall \widetilde{v}_i \iff \int_D \widetilde{p}^{(int)} dV = \int_{\partial D} \widetilde{p}^{(ext)} dS, \forall \widetilde{v}_i$$

*Code snippet:*

```
# Define internal virtual power
Pint=(
  M_el*Dx(u[0],0)*Dx(v[0],0) +  #sigma_22*v_2,2
  G_el*Dx(u[1],0)*Dx(v[1],0)    #sigma_12*v_1,2
  )*dx
# Define external virtual power
Pext=dot(ti,v)*ds

# Solve the problem (does the FE formulation, matrix assembly and
    linear solve)
solve(Pint == Pext, sol, bc)
```

# Summary

- Basic ideas and intuition behind the Principle of Virtual Powers
- Equivalence between equilibrium and PVP
- Use in discrete systems
- Use in classical continuum mechanics
- Use in generalized continua
- Use in upscaling where scale separation is no more valid
- Use in Finite Elements (to be continued…)
- Do the exercises…

# References to start from…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*Thank you for your attention!*