ALERT Geomaterials Doctoral School 2019

Strain localization in geomaterials and regularization: rate-dependency, higher order continuum theories and multi-physics

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GeM
Institut de Recherche en Génie Civil et Mécanique

erc
European Research Council
Established by the European Commission

CoQuake.com

Objectives

- Understand fundamental notions related to bifurcation theory;
- Perform a bifurcation analysis using the first Lyapunov method and derive the conditions for strain localization under different constitutive assumptions and continua;
- Identify the dominant time and spatial scales in a class of problems;
- Draw qualitative conclusions regarding strain localization zone thickness and mesh dependency without cumbersome numerical analyses;
- Understand the added-value of viscoplasticity and Micromorphic continua such as the Cosserat and strain-gradient continua;
- Investigate the effect of multiphysics couplings on the localization of deformations.

Packages

```
from knowledge import tensor_calculus, odes, stability
from character import perseverance
from problems import challenging
```

Packages

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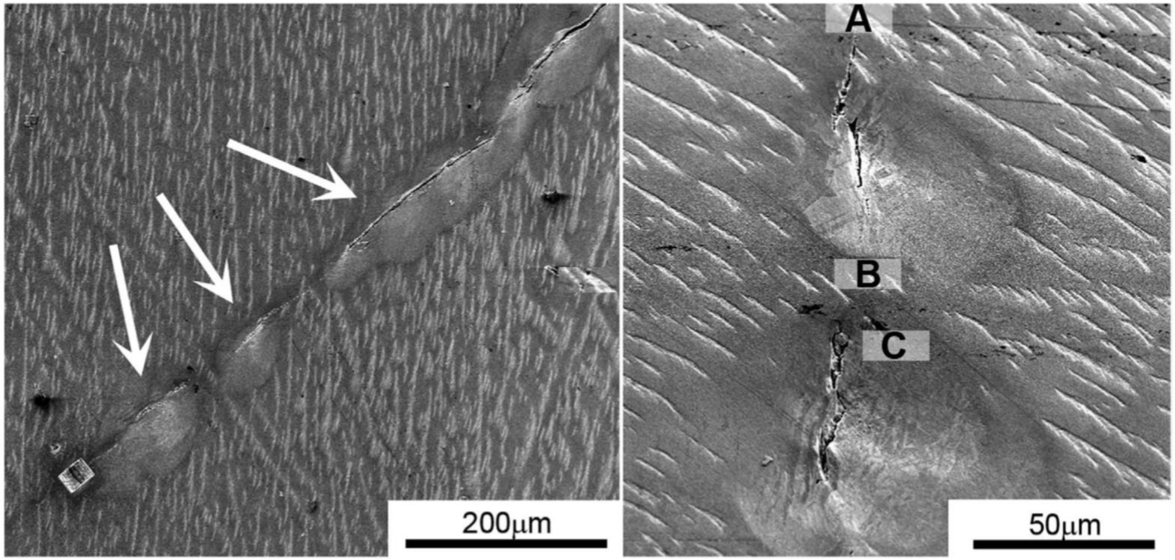
Chapter update

http://coquake.eu/index.php/tools/alert_2019/

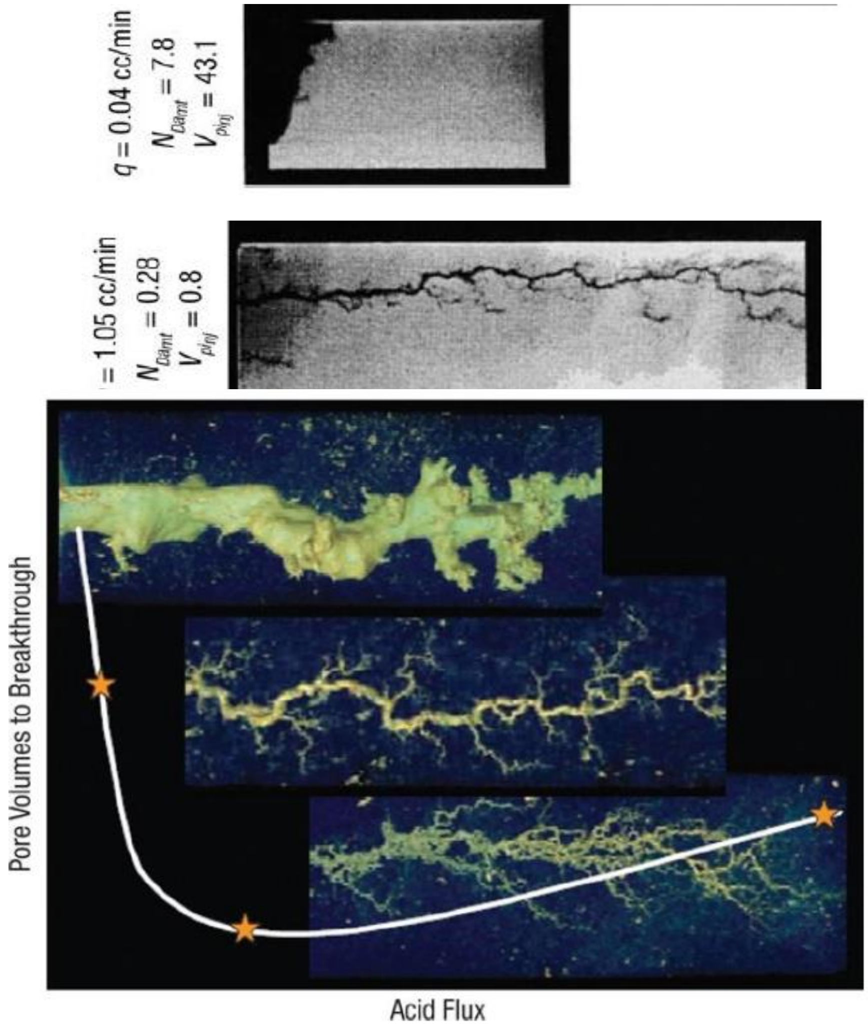
Examples of strain localization

Titanium after impact load



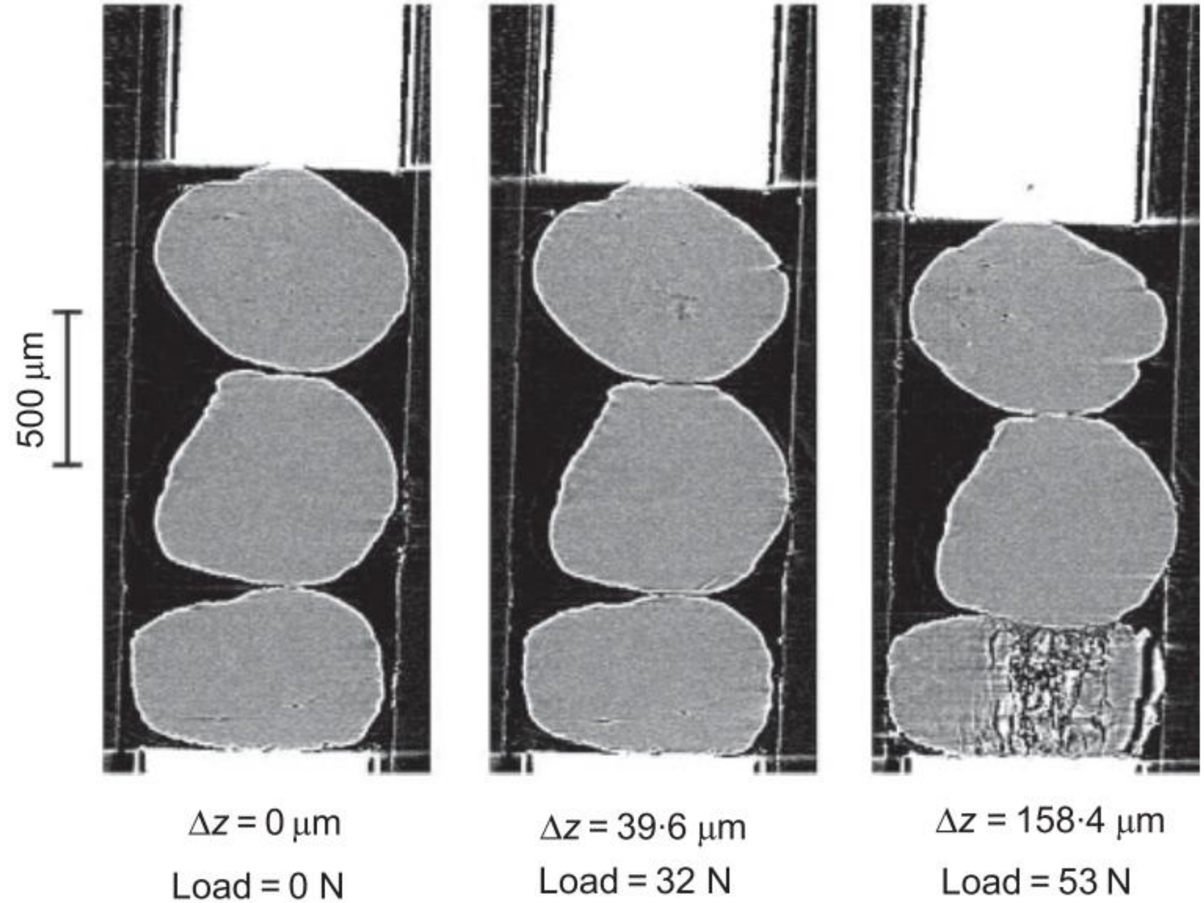
(P.Landau et al., Nature, 2016: “The genesis of adiabatic shear bands”)

Fingering with acidizing fluid in chalk



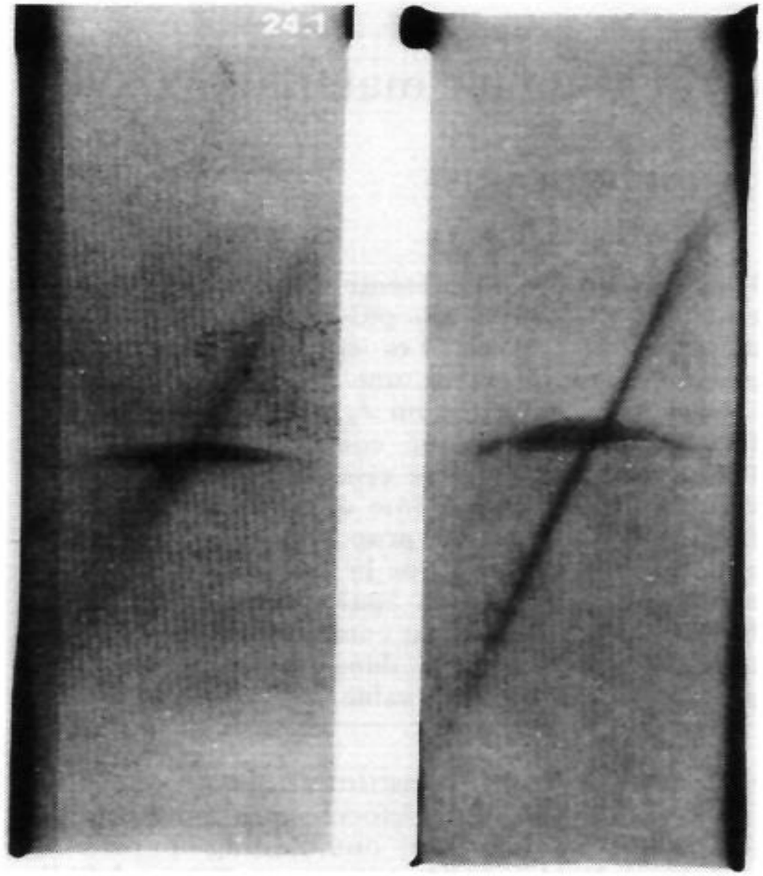
(Stroggylis & Papamichos, 2019)

Silica sand particles



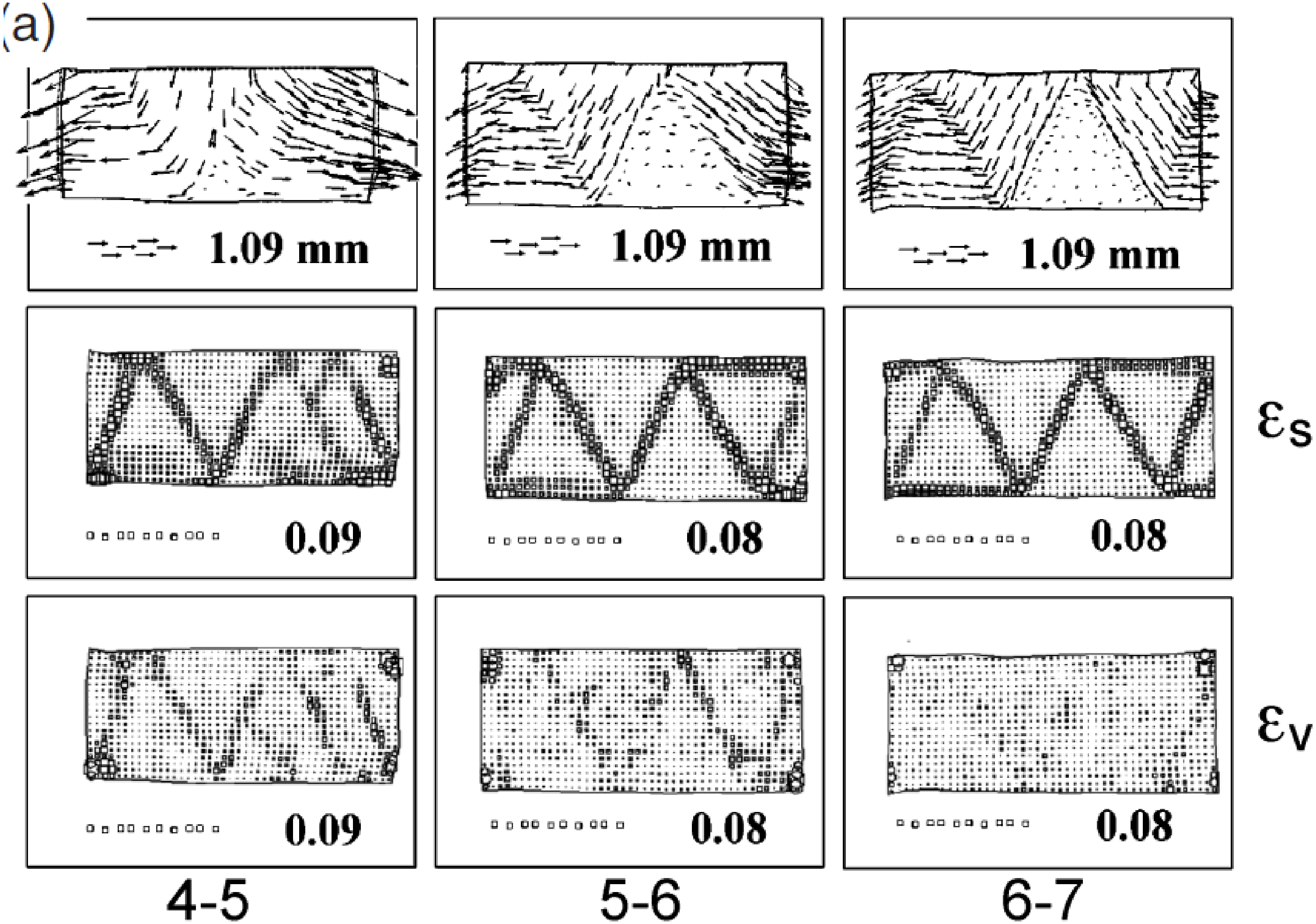
(Cil & Alshibli, 2012)

Biaxial tests



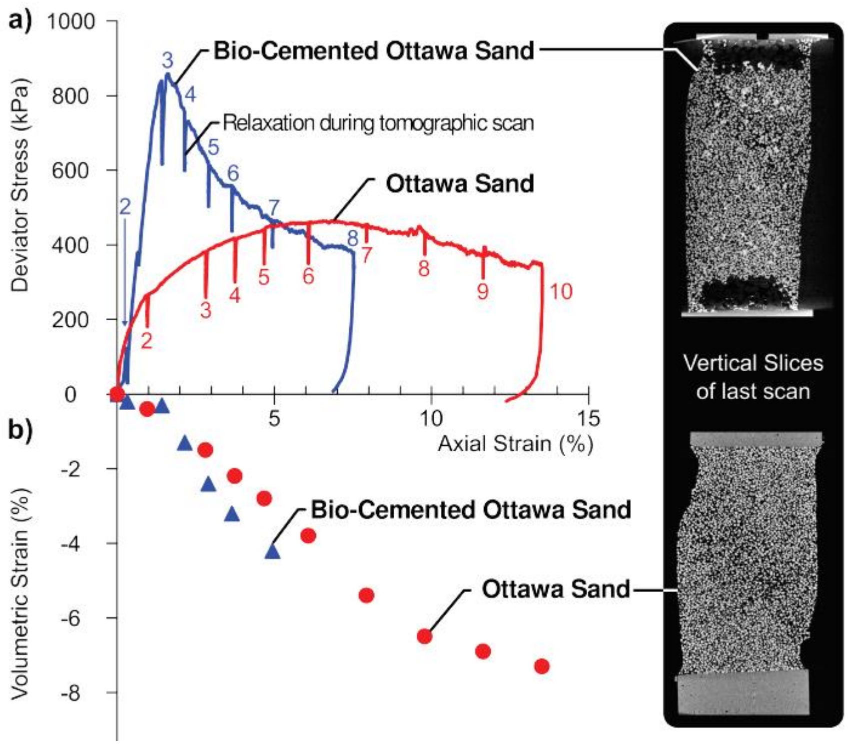
(Mühlhaus & Vardoulakis, 1987)

Shear tests



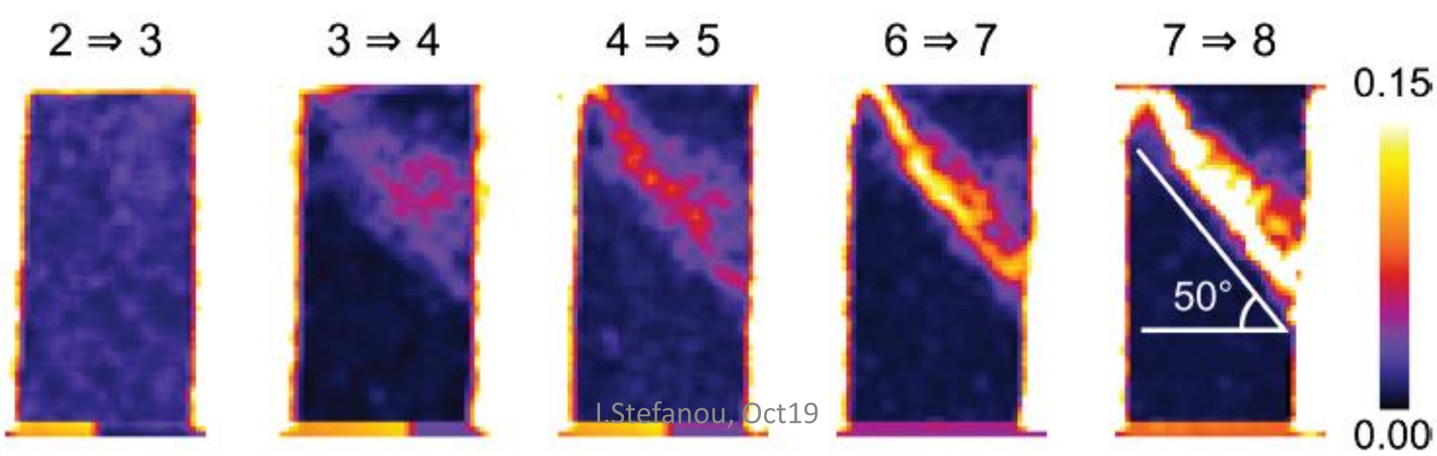
(Desrues & Viggiani, 2004)

Triaxial tests

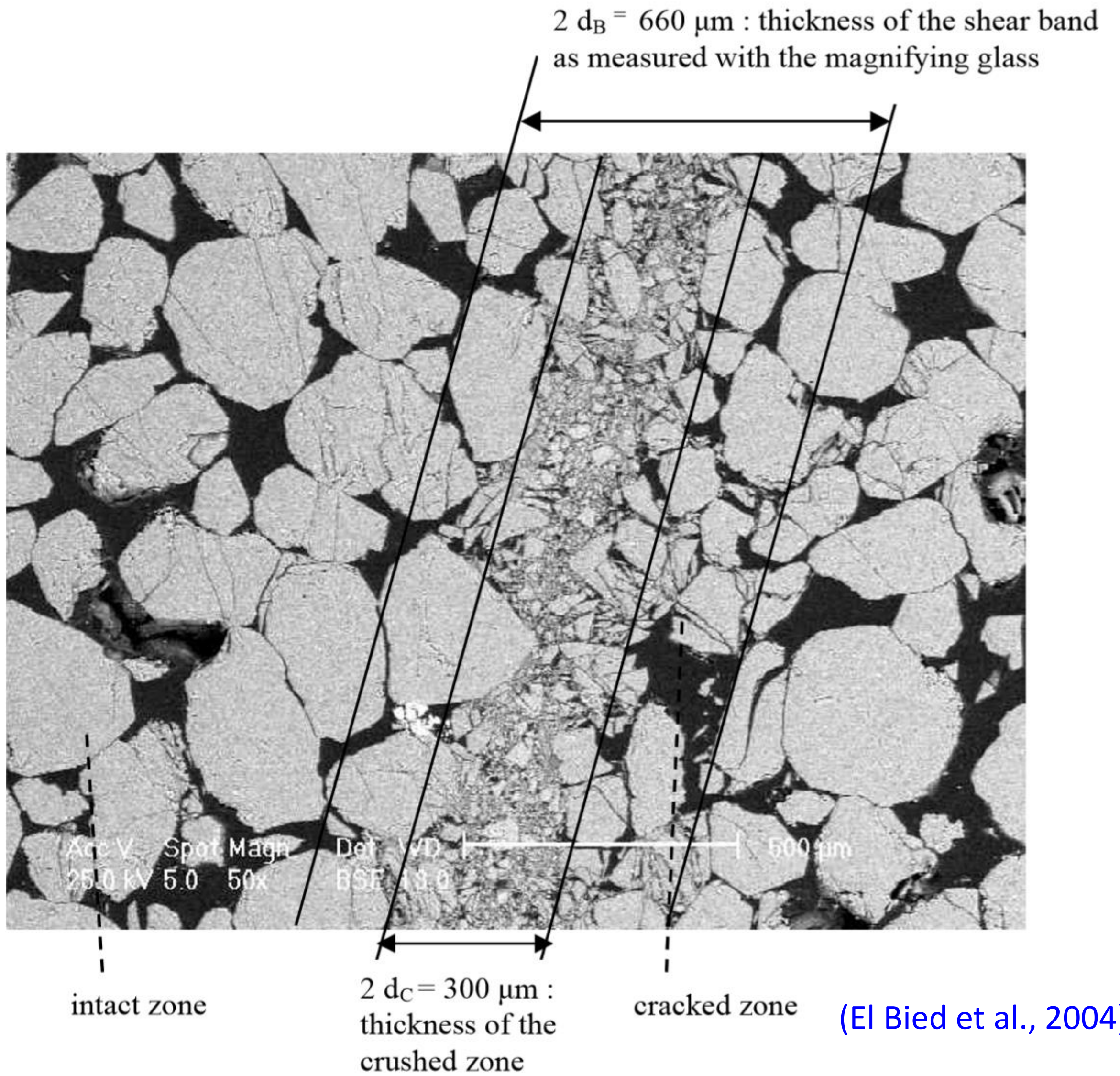


(Tagliaferri et al., 2017)

PIV

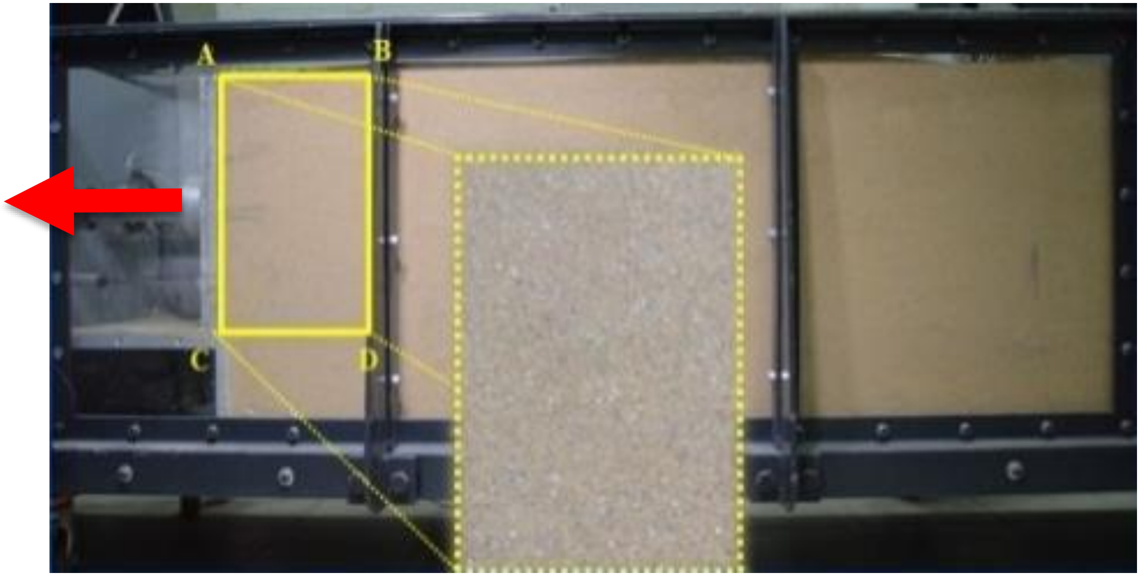


Zooming in…

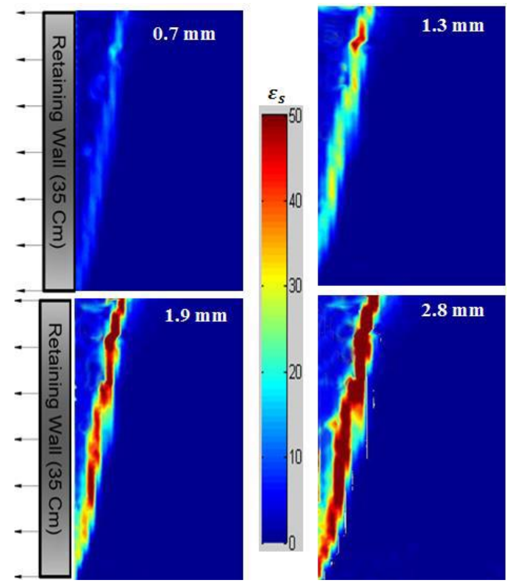


(El Bied et al., 2004)

Retaining walls



PIV

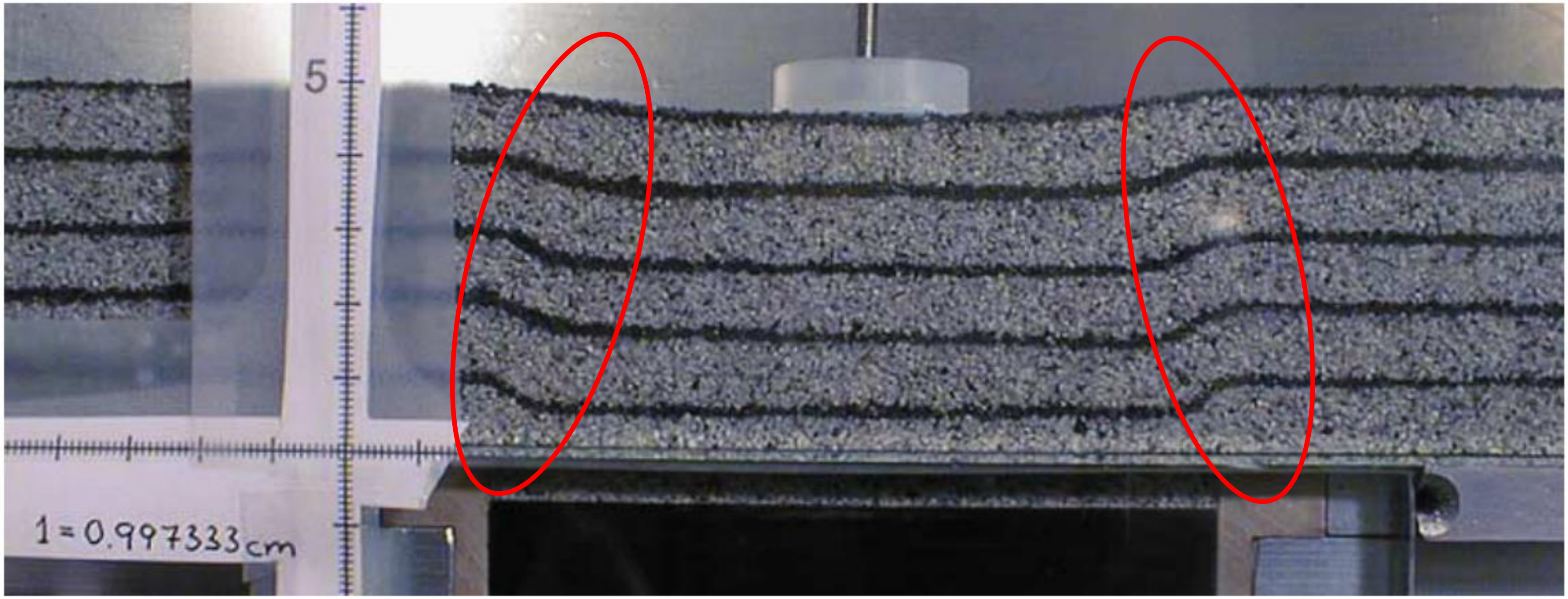


(Soltanbeigi et al., 2014)

Subsidence

(Vardoulakis et al., 2004)

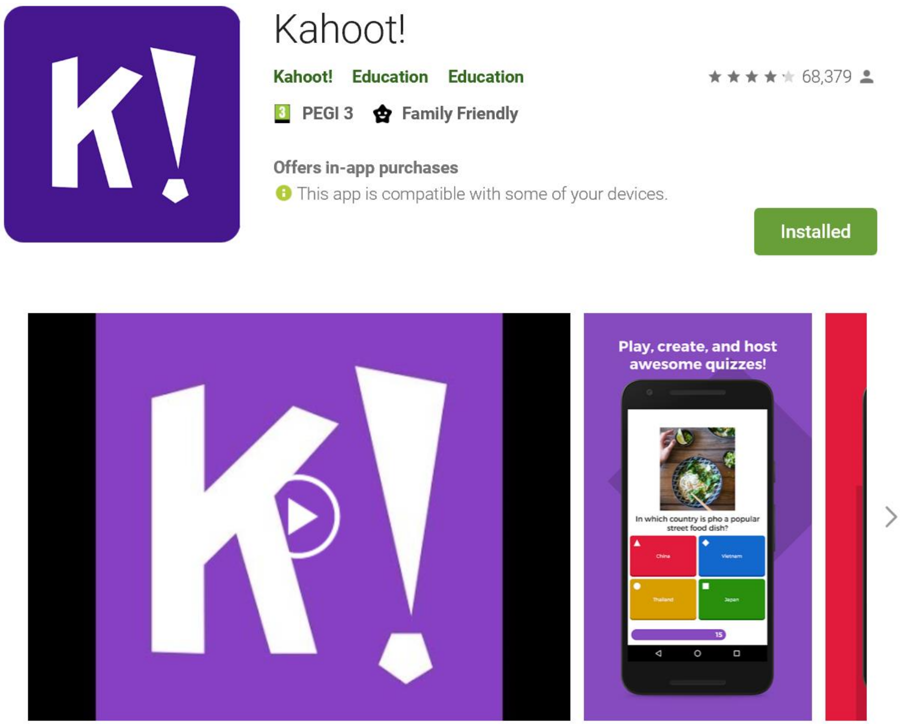
Subsidence



(Vardoulakis et al., 2004)

Fasten up!

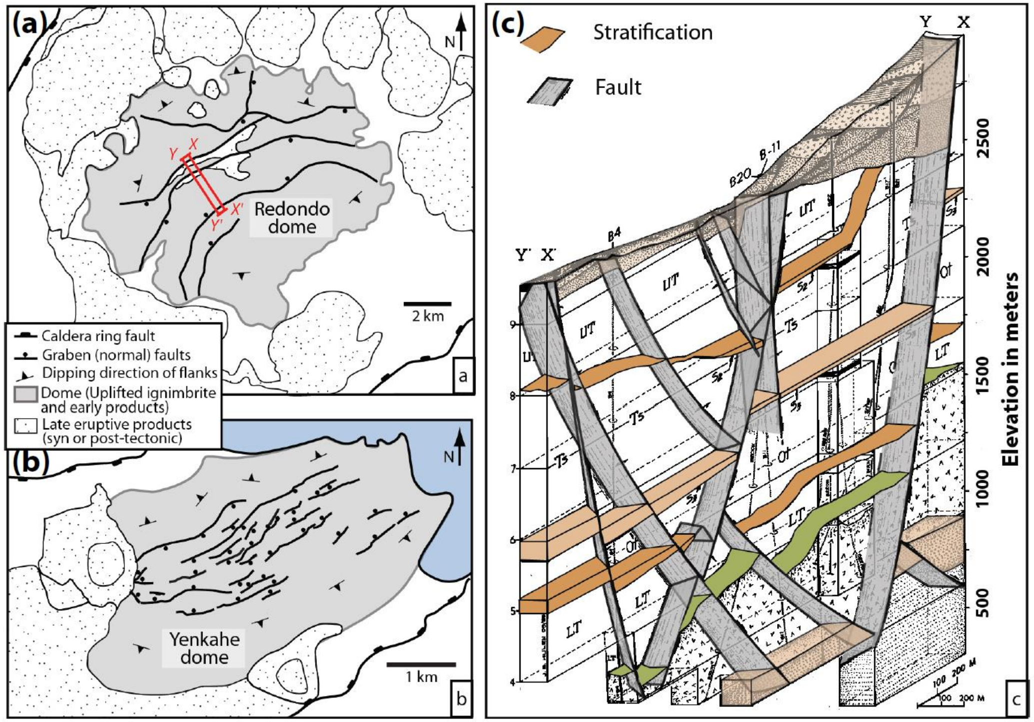
- Kahoot!   https://kahoot.com/



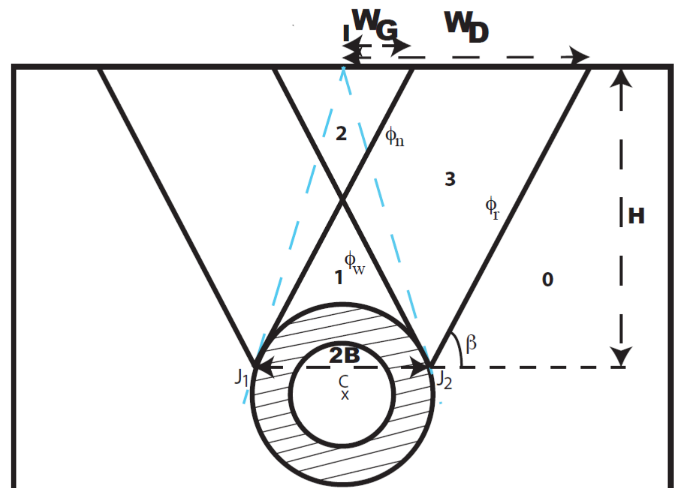
Q1-5

Compaction bands

Volcanoes



(a) Redondo dome in Valles caldera, NM US (Smith & Bailey, 1968)
(b) Yenkahe dome in Siwi caldera, IN (Brothelande et al., 2016)
(c) Valles caldera (Nielson & Hulen, 1984)



Fault pattern idealization
(J.L. Got et al., JGR, 2019)

Q6

EQ faults

San Andreas Fault (USGS)

Definitions

Q7-8

Loss of uniqueness

= existence of more than one (equilibrium or steady state) solutions

≠

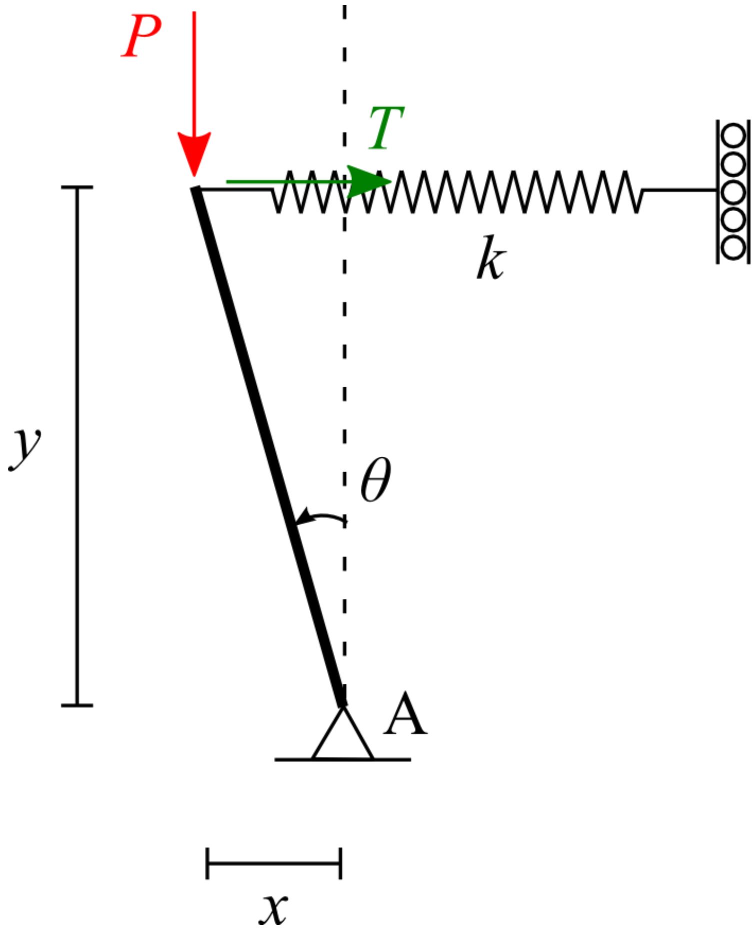
Bifurcation

≠

Instability

A simple system for building understanding

-> *Find all the equilibrium points (angles ϑ) of the system*:



A simple system for building understanding

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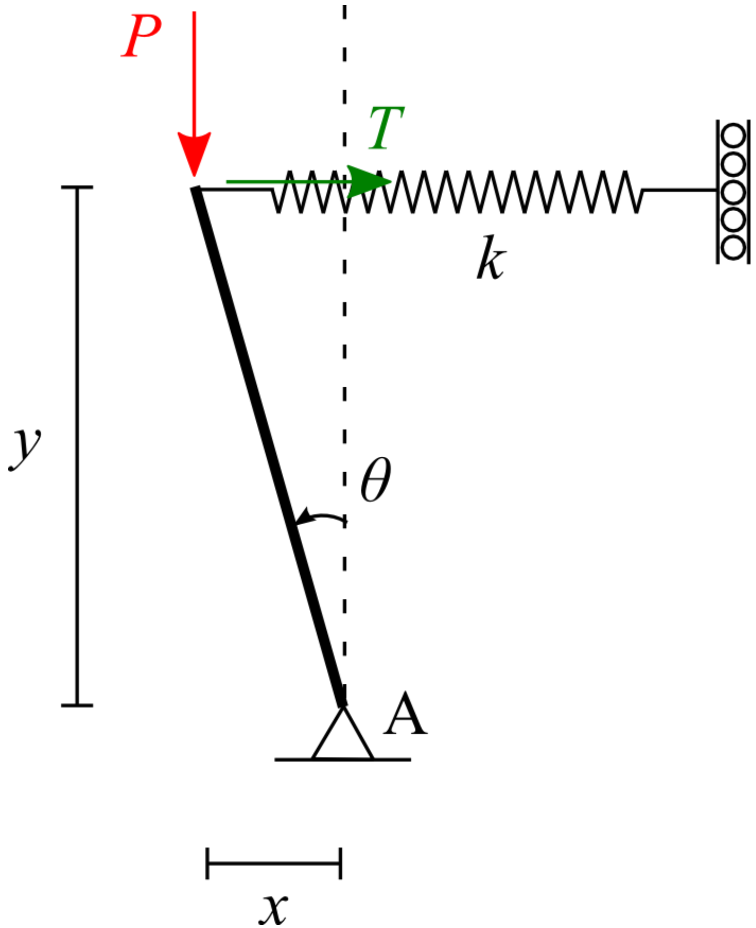


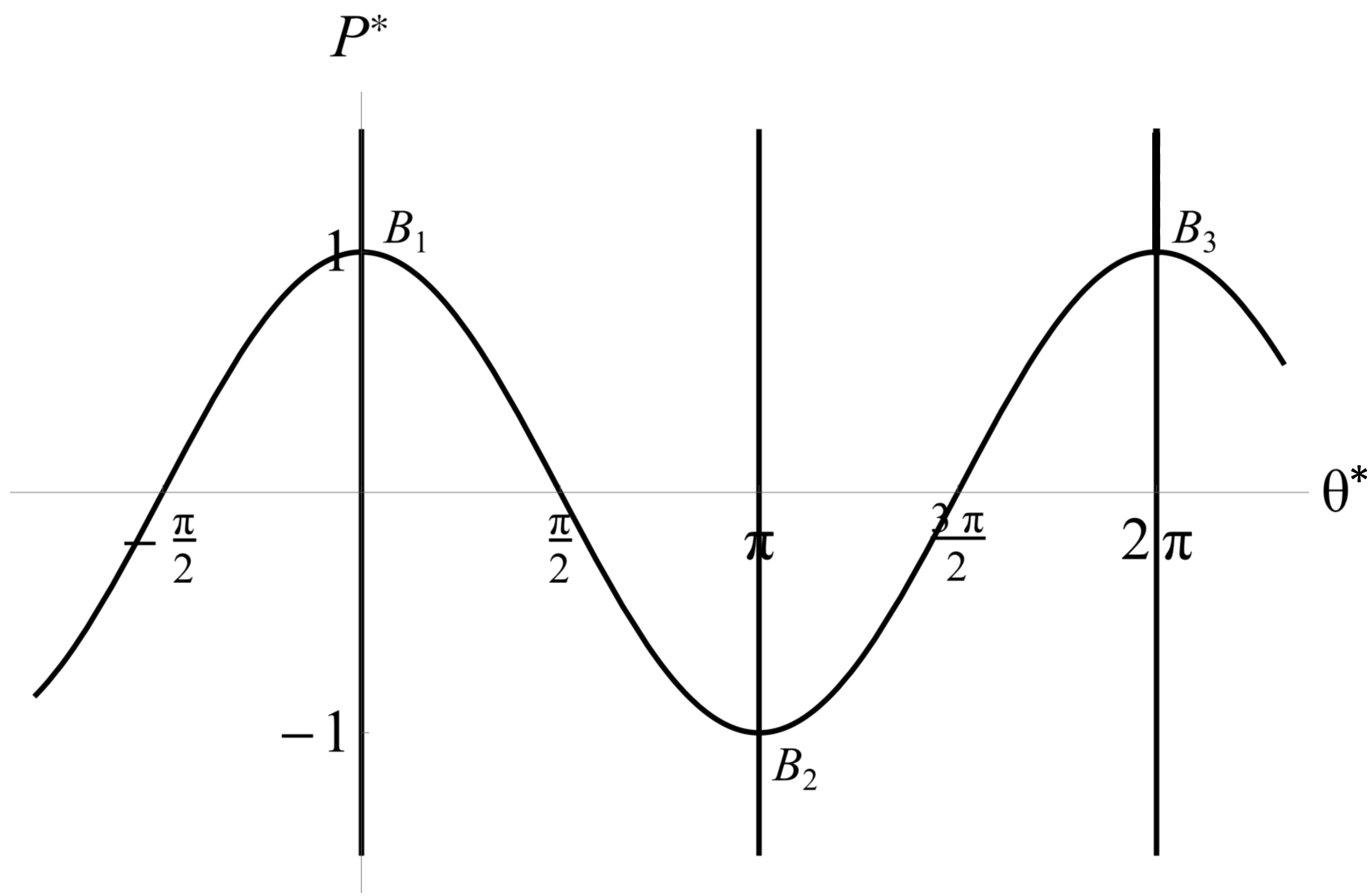
$$I_A \ddot{\theta} = \Sigma M_A = P\,x - T\,y$$

$$T = k\,x$$

$$x = \ell \sin\theta$$

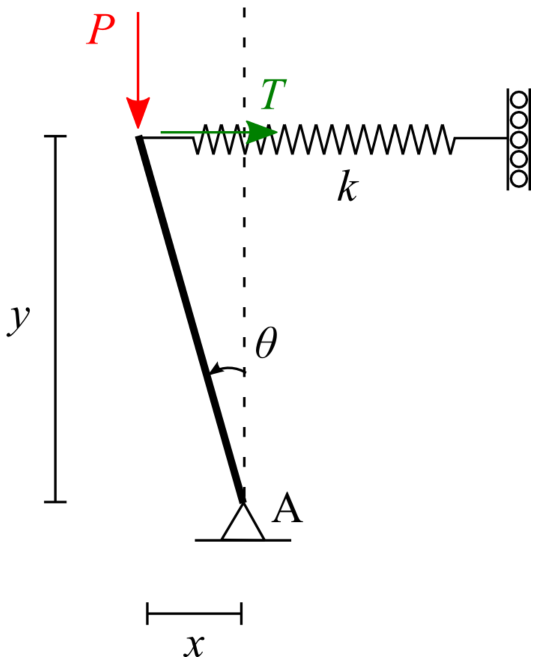
$$y = \ell \cos\theta$$

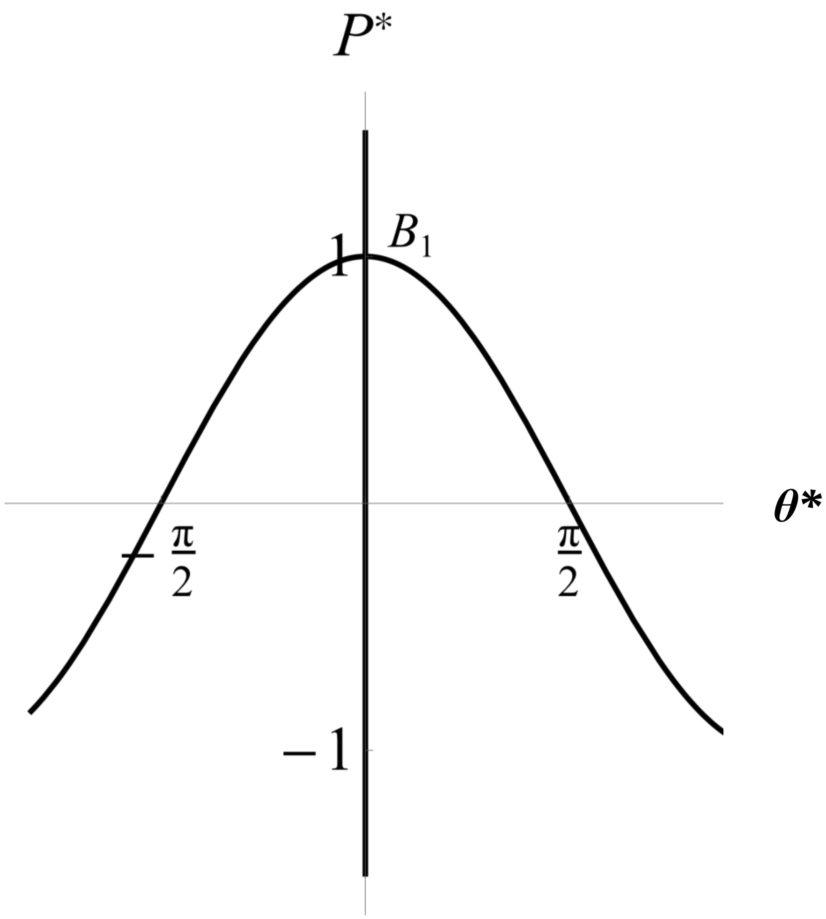
Equilibrium diagram



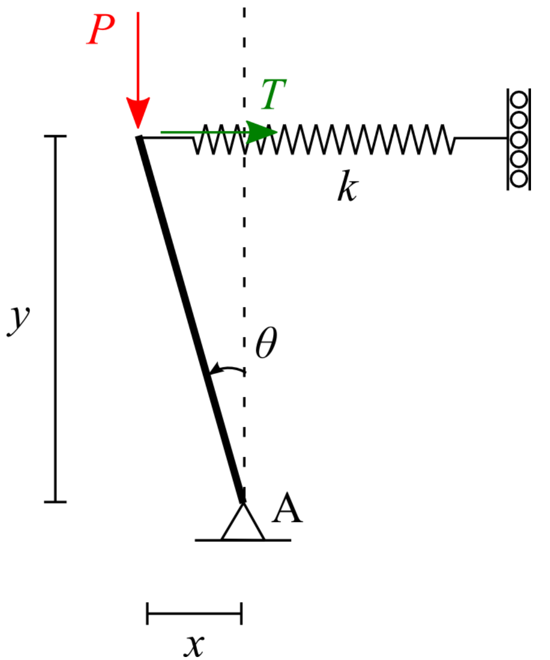
It is called also **bifurcation diagram** because at points B_1, B_2, B_3… the equilibrium diagram bifurcates!

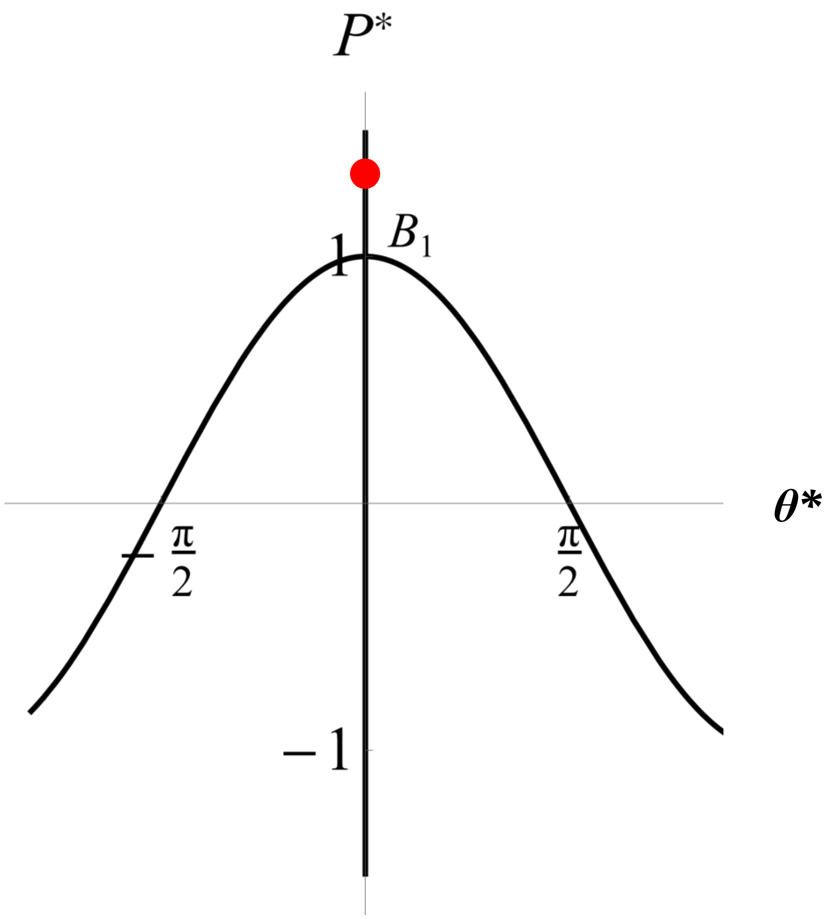
Loss of uniqueness



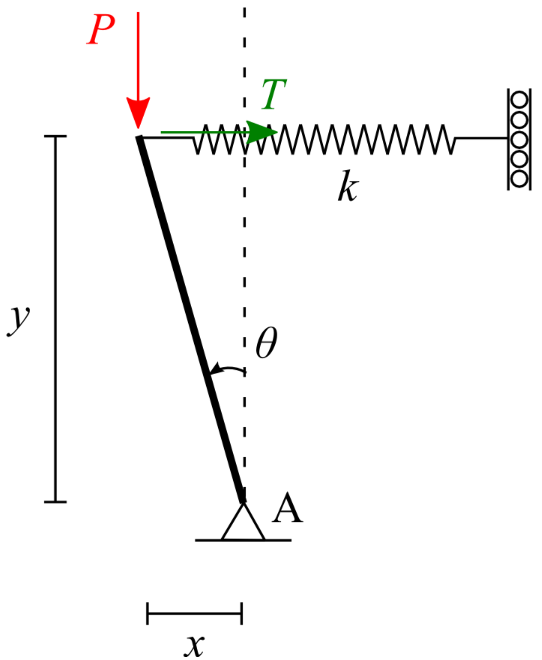


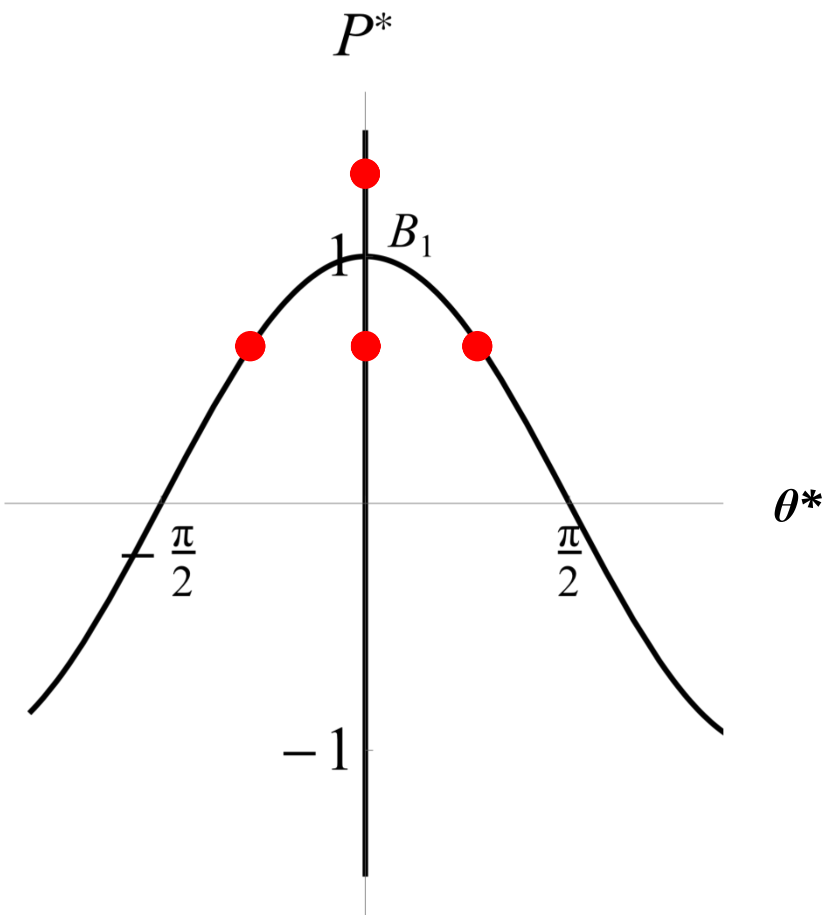
Loss of uniqueness



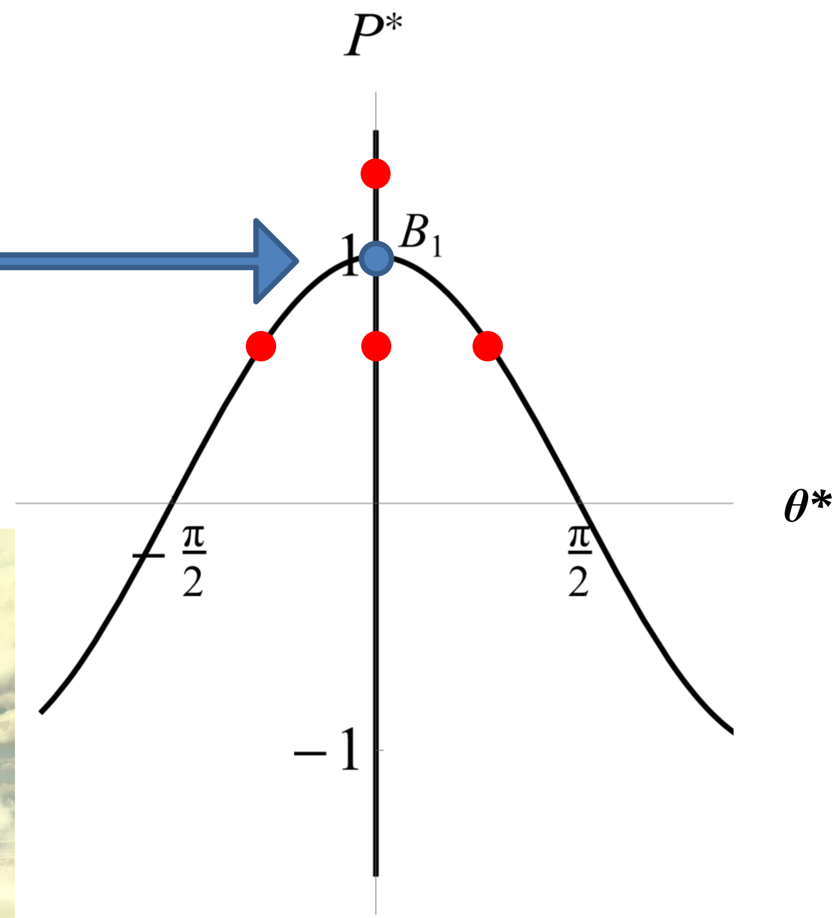


Loss of uniqueness

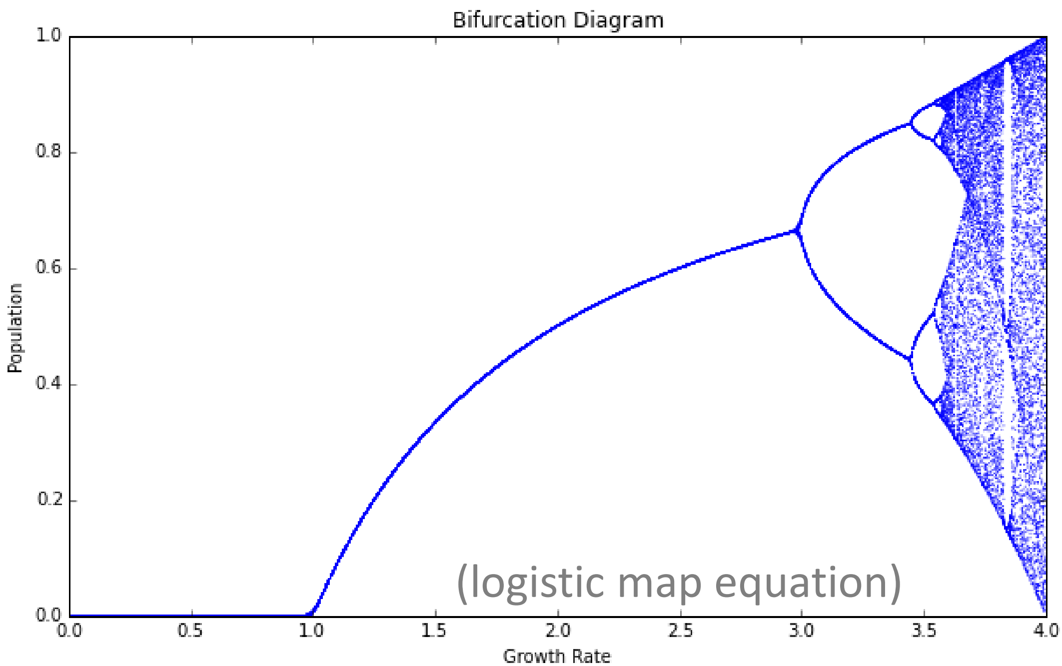




Loss of uniqueness

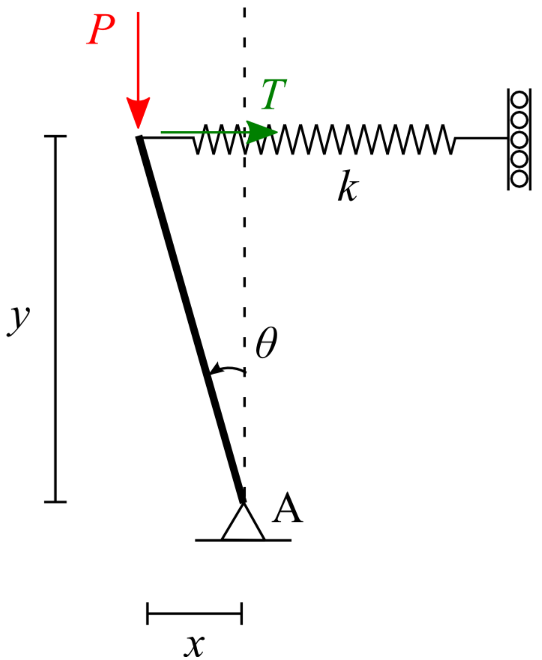


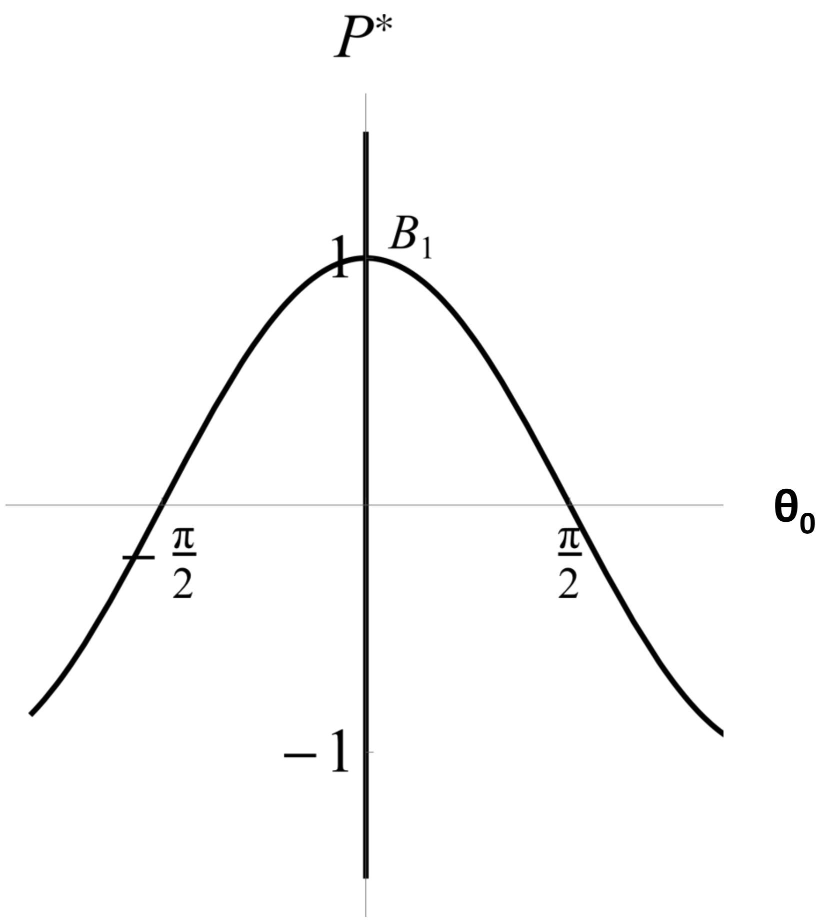
Bifurcation point



It might be simple or complicated… but the idea is the same.

How does the system decide where to go?





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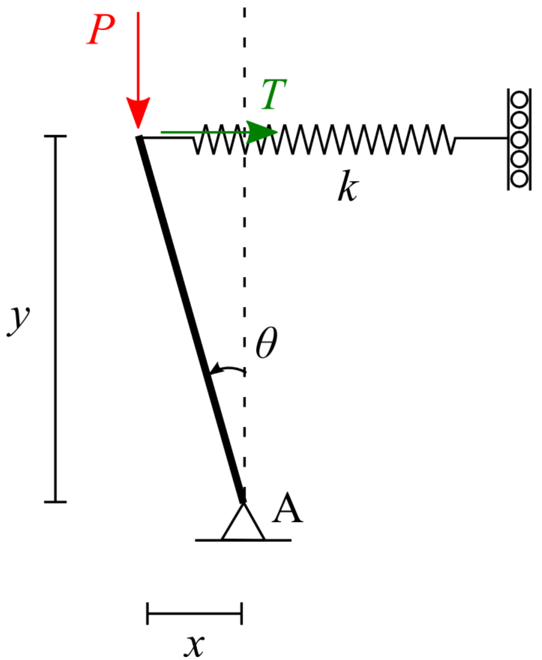

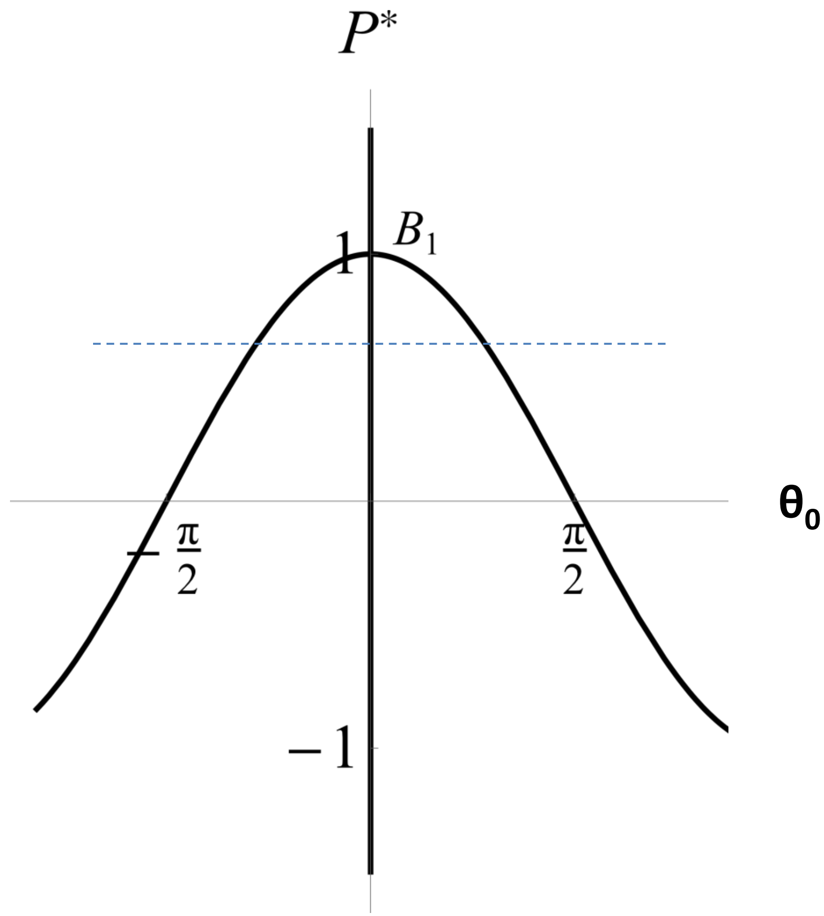

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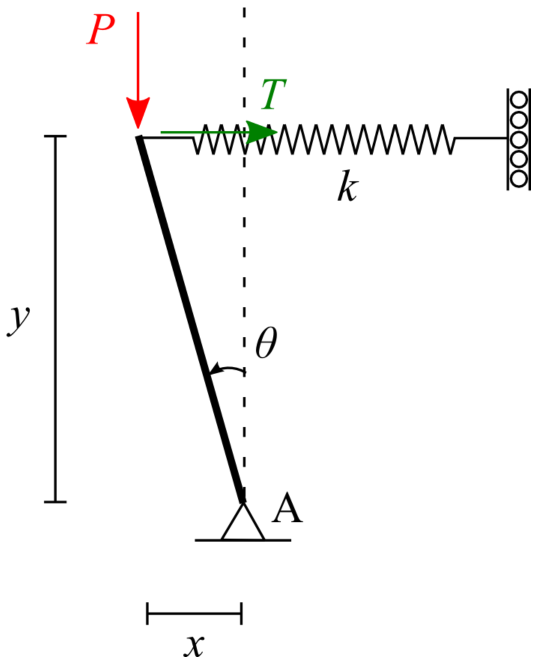

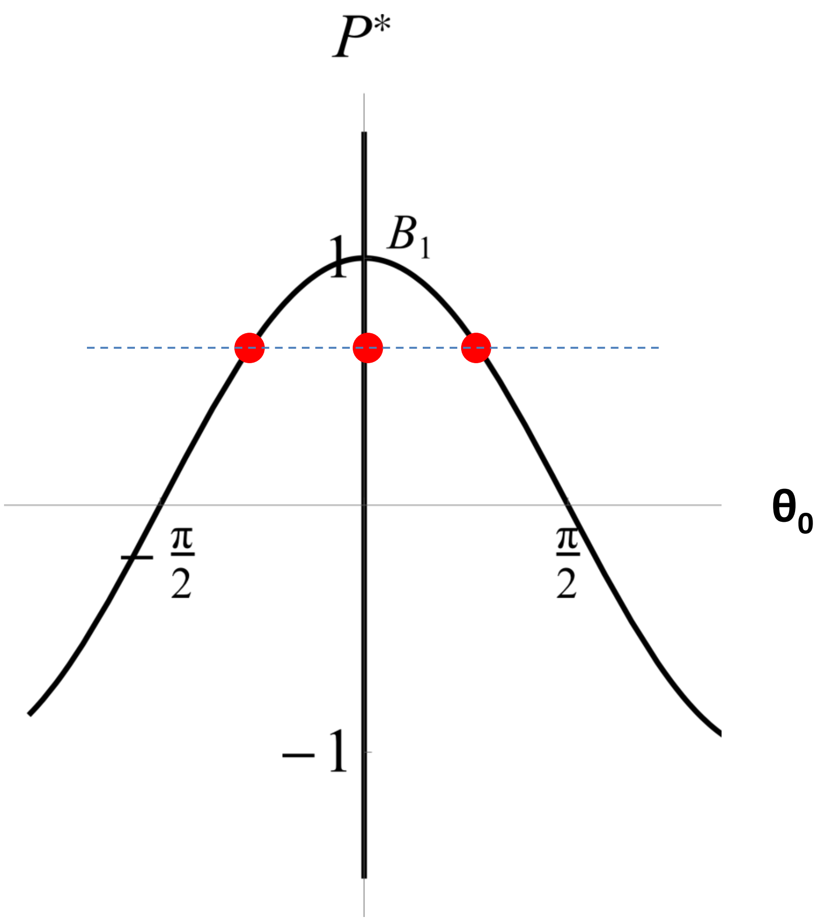

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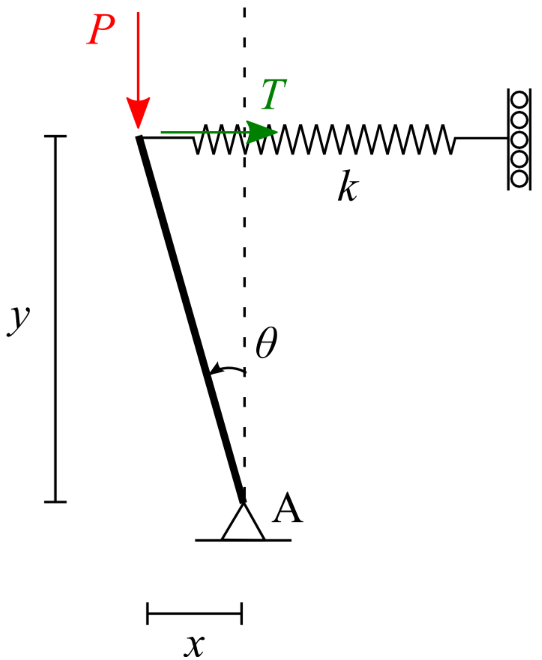
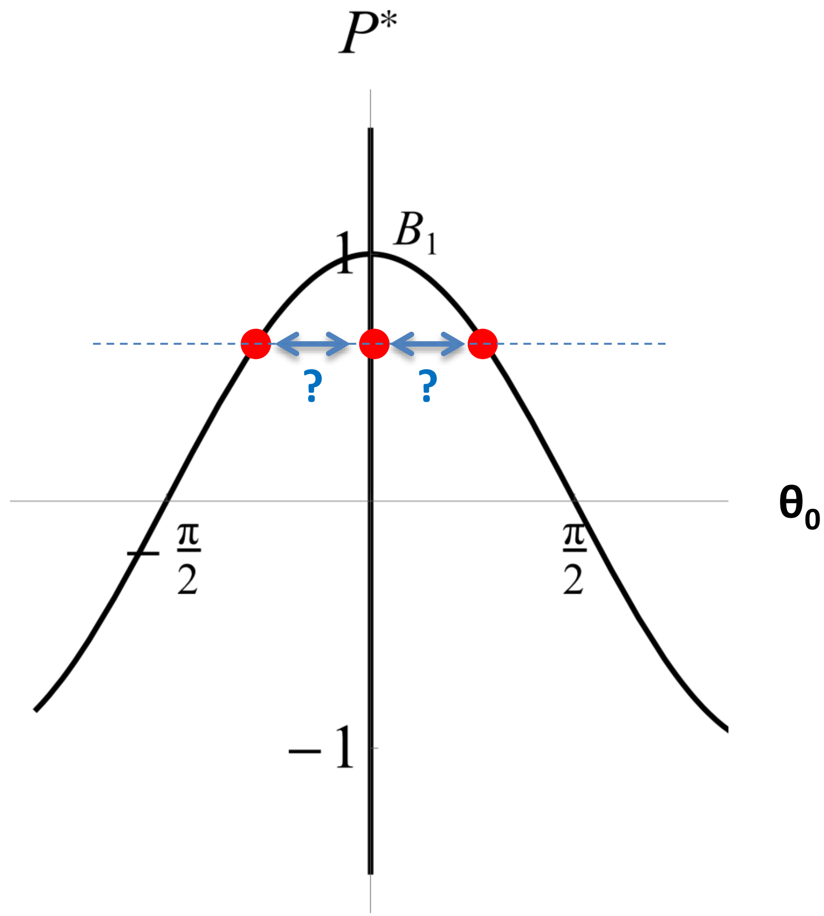
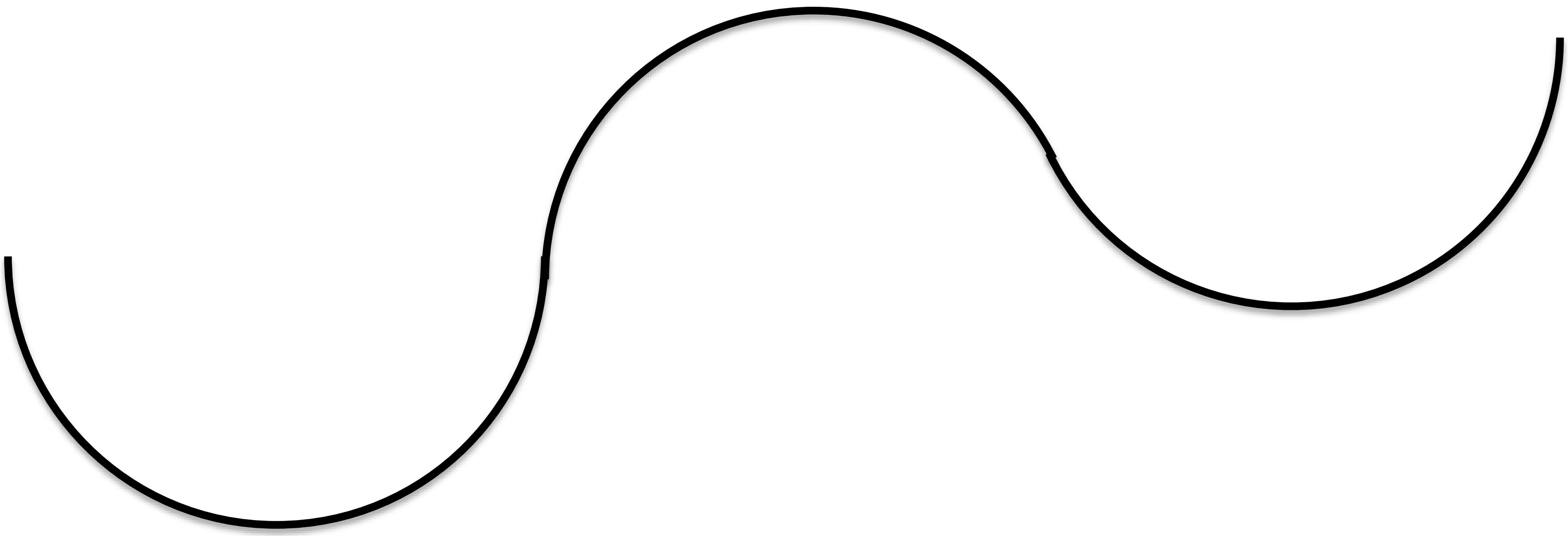
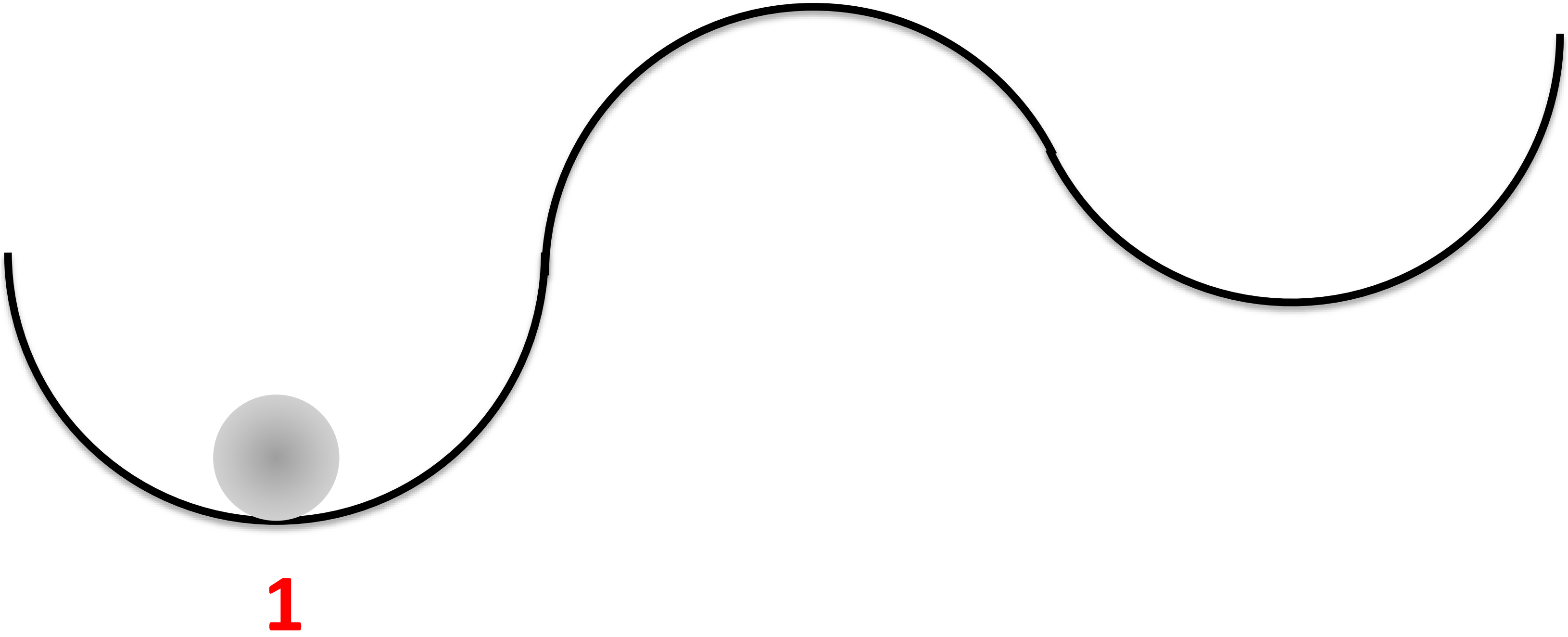
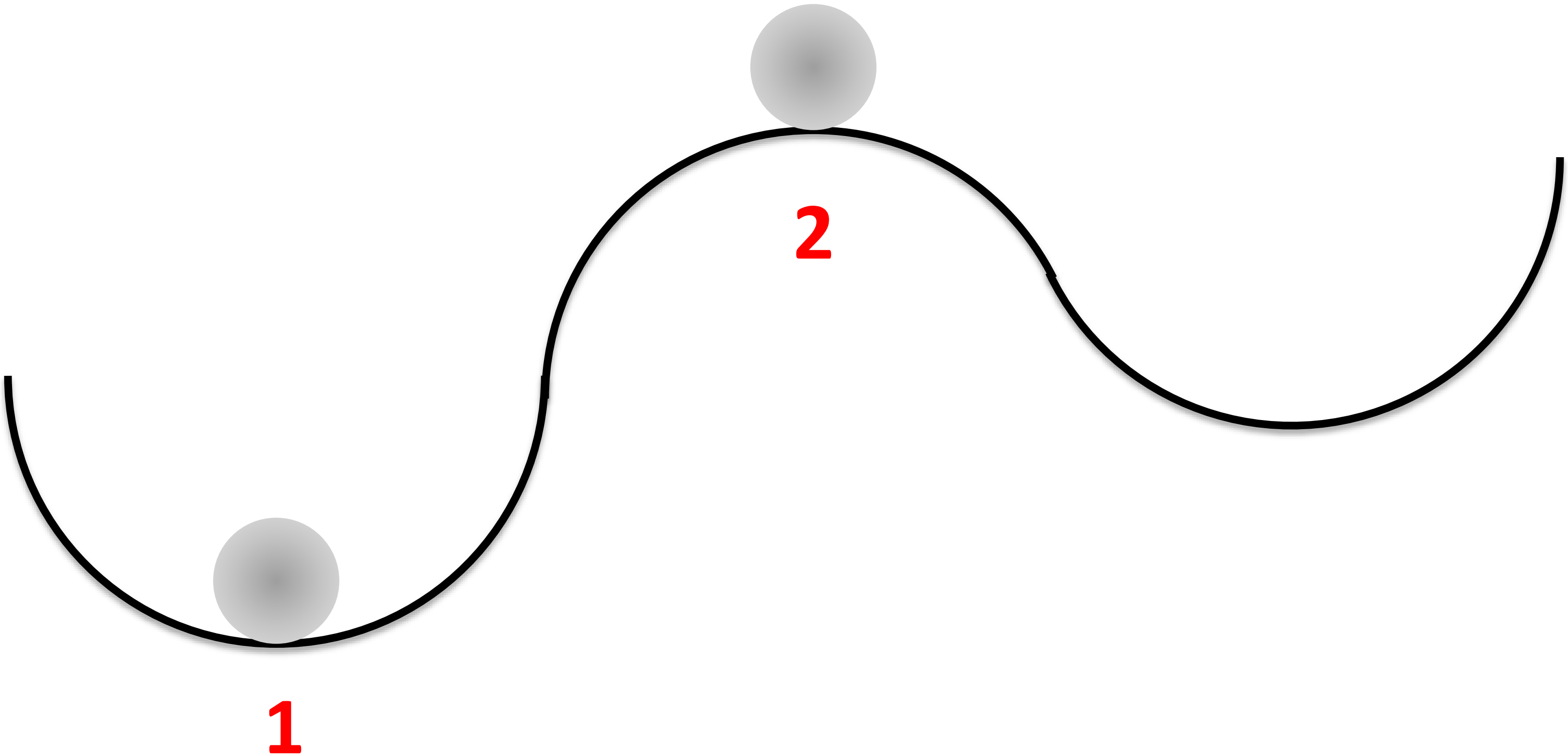
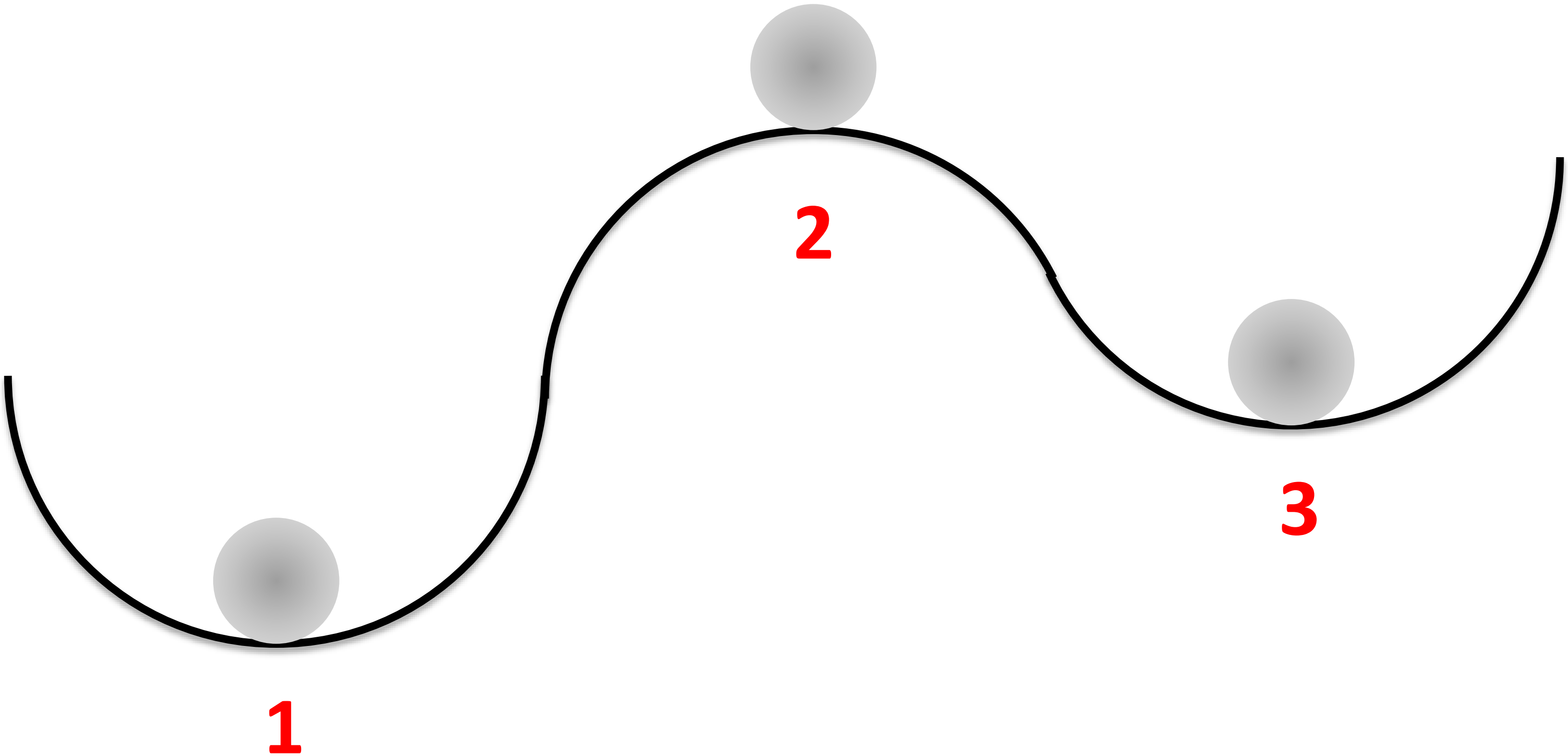


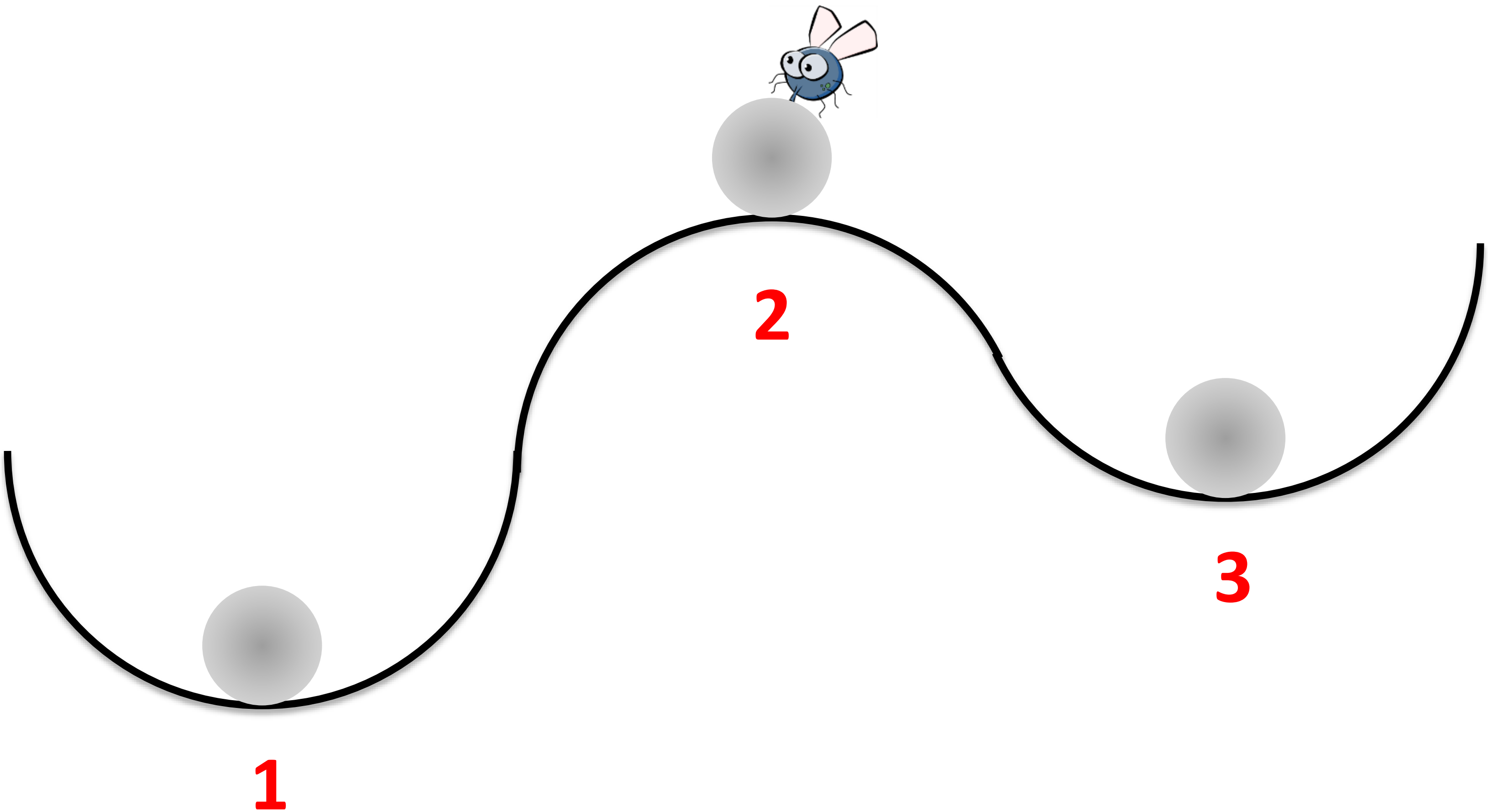


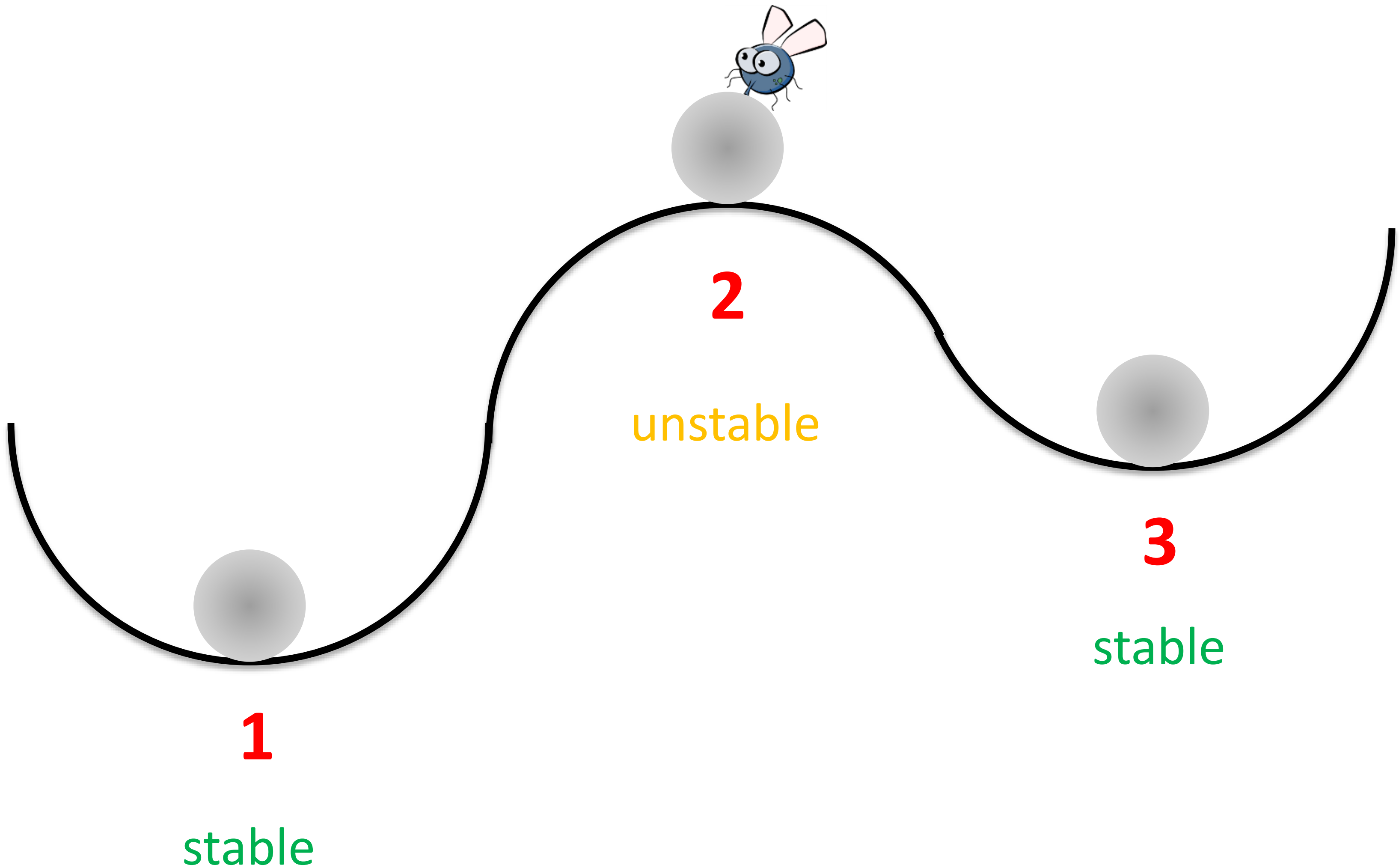



1


2
1


2
3
1


2
3
1


2
unstable
3
stable
1
stable

The notion of (Lyapunov) stability

If we apply a **small perturbation** (the fly!) and the system **stays close or returns** back to its equilibrium

↓

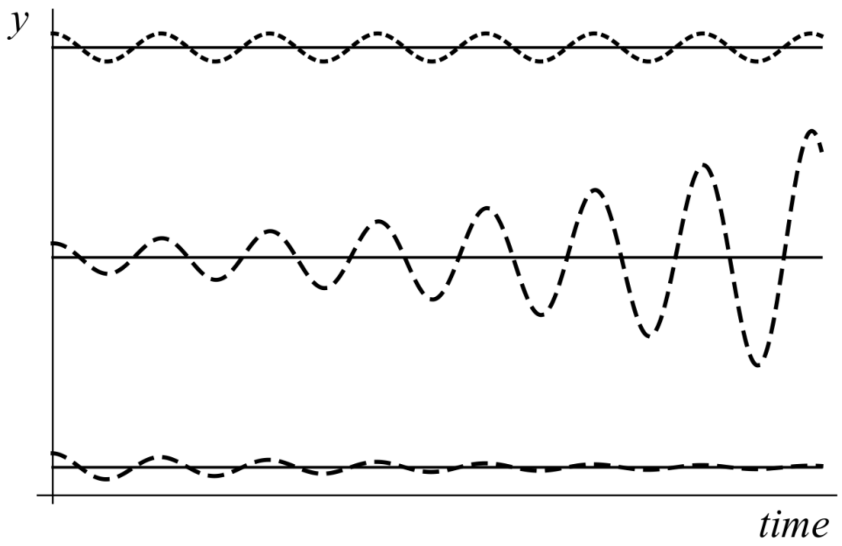
Stable equilibrium

If we apply a small perturbation and the system **moves away** from its equilibrium

↓

Unstable equilibrium

Stability theory was formulated in 1892 by A.M.Lyapunov (1857-1917).



Time… is central even if we forget it or don't consider it directly in our analyses.

Other stability postulates

- Second order work
- Hill's stability
- Mandel's stability
- Loss of ellipticity
- Loss of controllability
- …

Other stability postulates

- Second order work
- Hill's stability
- Mandel's stability
- Loss of ellipticity
- Loss of controllability
- …

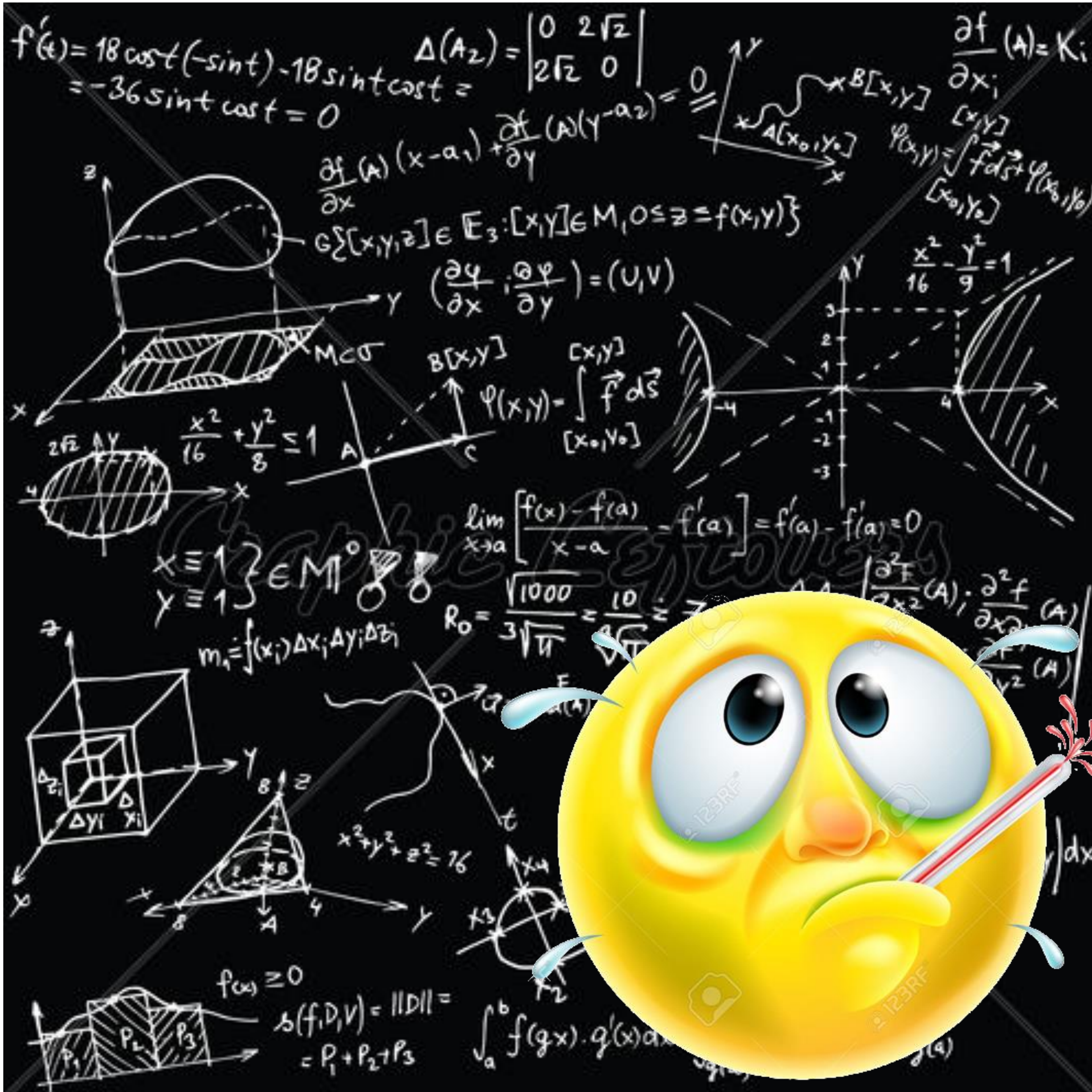
Confused?

A couple of nice papers that **clarify** the applicability of many other than Lyapunov stability postulates are:

Chambon, R., D. Caillerie, and G. Viggiani (2004), Loss of uniqueness and bifurcation vs instability: some remarks, *Rev. Française Génie Civ.*, *8*(5–6), 517–535.
(ALERT School 2004)

Bigoni, D., and T. Hueckel (1991), Uniqueness and localization—I. Associative and non-associative elastoplasticity, *Int. J. Solids Struct.*, *28*(2), 197–213.

Q9



Mathematical problem

Well-posedness

1) A solution exists;

2) The solution is unique;

3) The solution's behavior changes continuously with the initial conditions

Jacques Hadamard. Sur les problèmes aux dérivées partielles et leur signification physique. Princeton University Bulletin, 49-52, 1902.

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3) The solution's behavior changes continuously with the initial conditions

Jacques Hadamard. Sur les problèmes aux dérivées partielles et leur signification physique. Princeton University Bulletin, 49-52, 1902.

We say that a problem is ill-posed when it is not well-posed.

Study of strain localization

Concept

Is the homogeneous deformation of a solid, stable?

Instability of homogeneous deformation

Dynamic equations
of a Cauchy continuum: $\sigma_{ij,j} = \rho \ddot{u}_i$

Equilibrium point: $\sigma^*_{ij,j} = 0$

Let's assume that we are in a state of homogeneous deformation (everywhere the same and constant).

>>> We want to investigate the possibility of non-homogeneous deformations such as compaction, shear and dilation bands.

Example:

Successive equilibria for increasing *P*:

$$\sigma^{*}_{ij,j} = 0$$

Considering the class of materials that σ can be linearized (hypothesis of equivalent material/linear comparison solid):

$$\sigma_{ij} = \sigma_{ij}^{*} + \tilde{\sigma}_{ij} = \sigma_{ij}^{*} + L_{ijkl}\tilde{u}_{k,l}$$

(Rice, 1976)

$\tilde{u}_i$ is a perturbation from the reference, homogeneous, equilibrium configuration u_i^*, such that: $\tilde{u}_i = u_i - u_i^*$

Replacing: $$L_{ijkl}\tilde{u}_{k,lj} = \rho \ddot{\tilde{u}}_i$$

$$L_{ijkl}\tilde{u}_{k,lj} = \rho\ddot{\tilde{u}}_i$$

$$L_{ijkl}\tilde{u}_{k,lj} = \rho\ddot{\tilde{u}}_i$$

Separation of variables:

$$\tilde{u}_i = X(x_p)U_i(t) \qquad \rightarrow \qquad L_{ijkl}X_{,lj}\,U_k(t) = \rho X\,\ddot{U}_i(t)$$

$$L_{ijkl}\tilde{u}_{k,lj} = \rho \ddot{\tilde{u}}_i$$

Separation of variables:

$$\tilde{u}_i = X(x_p)U_i(t) \qquad \rightarrow \qquad L_{ijkl}X_{,lj}\,U_k(t) = \rho X\,\ddot{U}_i(t)$$

General solution in time:

$$U_i(t) = g_i e^{st}$$

$$L_{ijkl}\tilde{u}_{k,lj} = \rho\ddot{\tilde{u}}_i$$

Separation of variables:

$$\tilde{u}_i = X(x_p)U_i(t) \qquad \rightarrow \qquad L_{ijkl}X_{,lj}\,U_k(t) = \rho X\,\ddot{U}_i(t)$$

General solution in time:

$$U_i(t) = g_i e^{st}$$

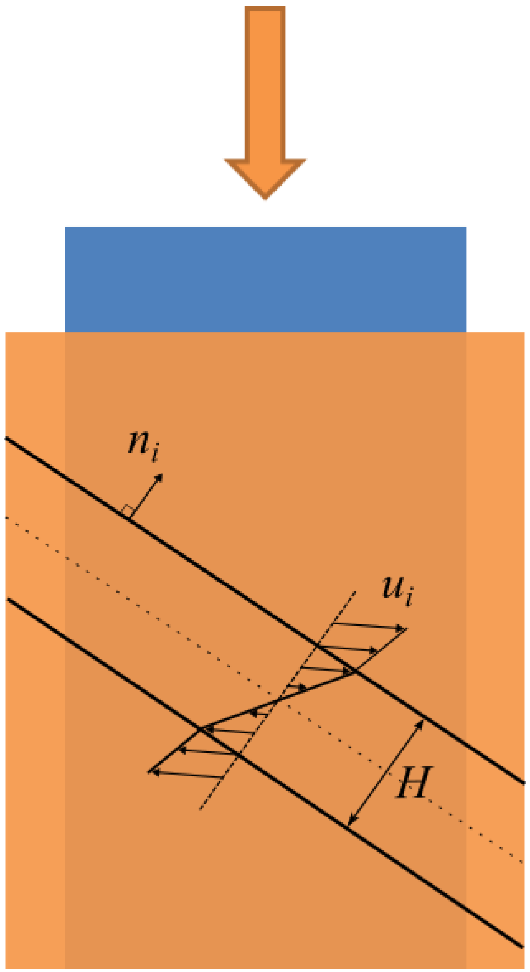
Leading to: $$\left(L_{ijkl}X_{,lj} - \rho X s^2\delta_{ik}\right) g_k = 0$$

Deformation bands

Allowing plane wave solutions for

X that satisfy the *BC's*

$$X(x_p) = e^{i\frac{2\pi}{\lambda}n_p x_p}$$



$$\left[\Gamma_{ik} + \rho\left(\frac{\lambda s}{2\pi}\right)^2 \delta_{ik}\right] g_k = 0$$

$$\Gamma_{ik} = n_j L_{ijkl} n_l \quad \text{(\textit{acoustic tensor})}$$

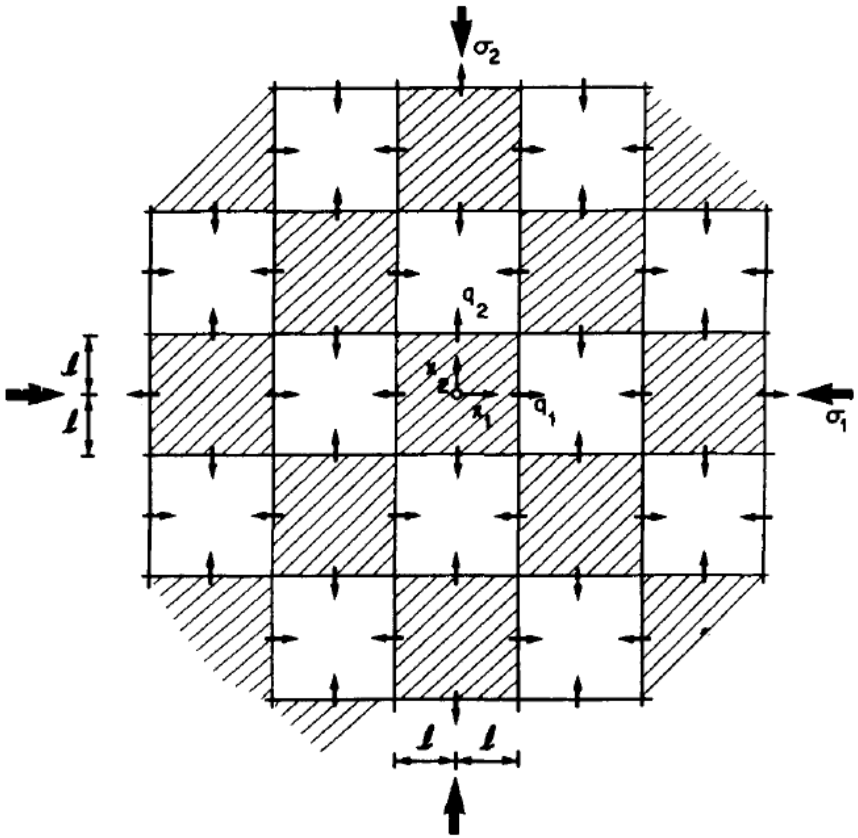
If $\rho\left(\frac{\lambda s}{2\pi}\right)^2 = something > 0 \;\Rightarrow Re(s) > 0$ then

the homogeneous solution is **unstable and the system will bifurcate to a non-uniform solution** (which we do not need to find).

The above condition is independent of the specific constitutive law, provided that it is rate-independent.

Q10

Q10



(Vardoulakis & Sulem, 1995)

Types of deformation bands

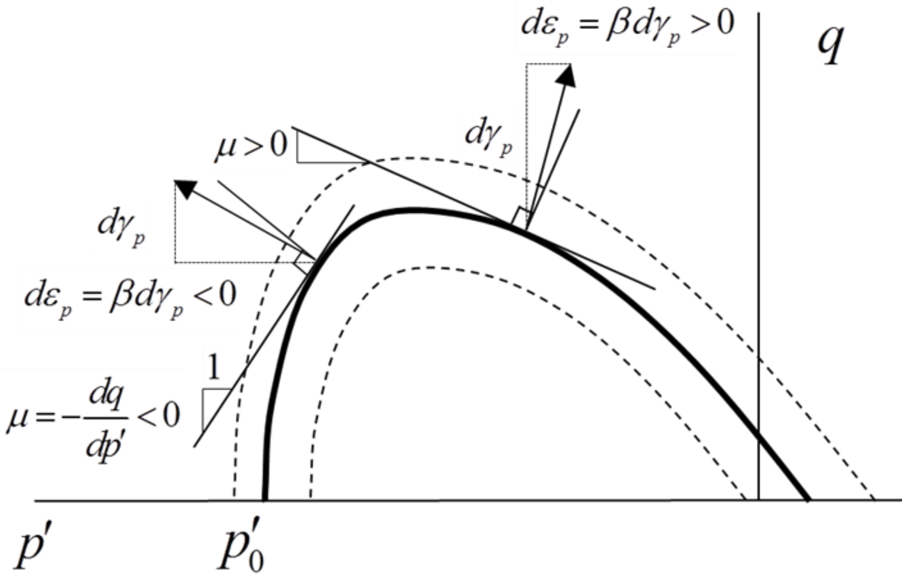
The type of the deformation band (compaction, shear or dilation band) is determined by the product $g_i n_i$.

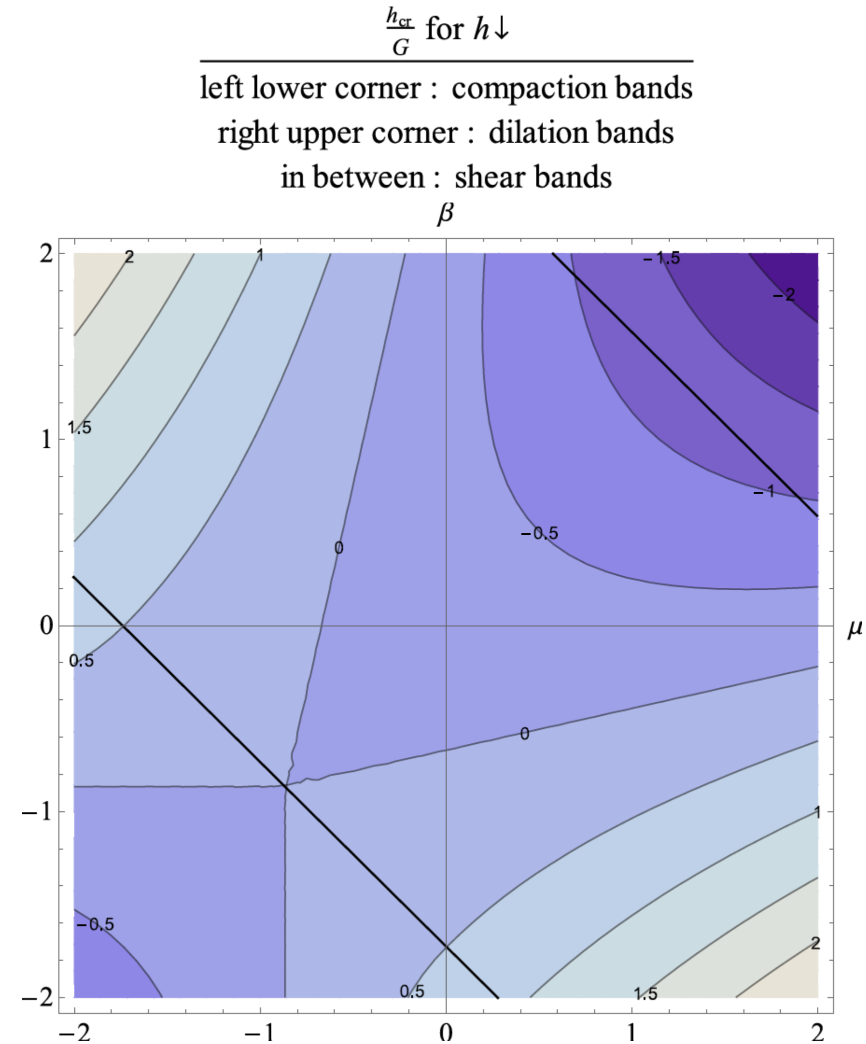
$g_i n_i = -1$ pure compaction band

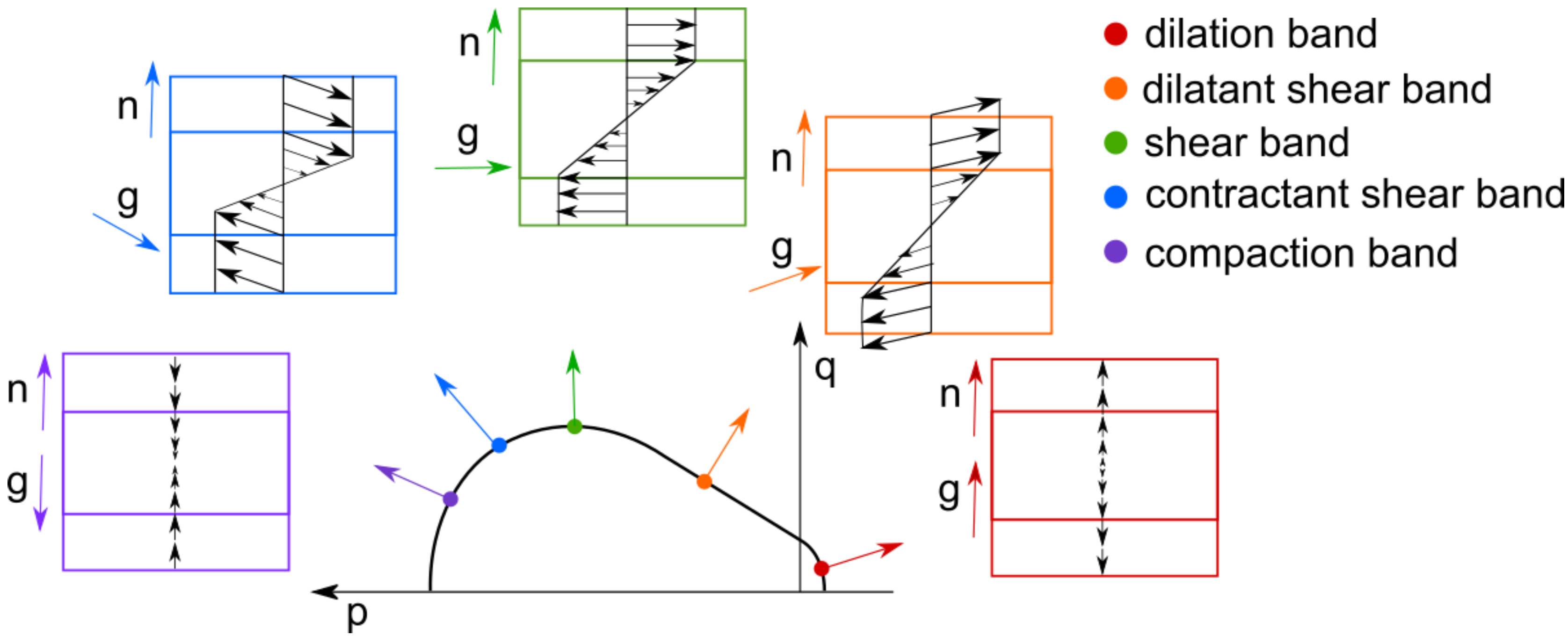
$g_i n_i = 0$ shear band

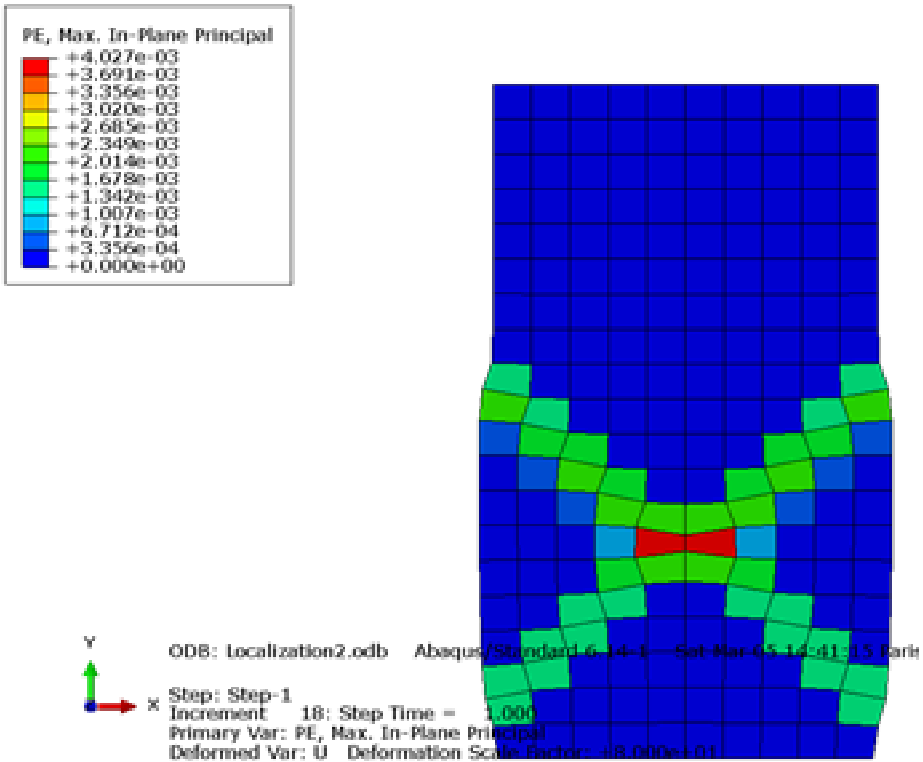
$g_i n_i = +1$ pure dilation band

Example

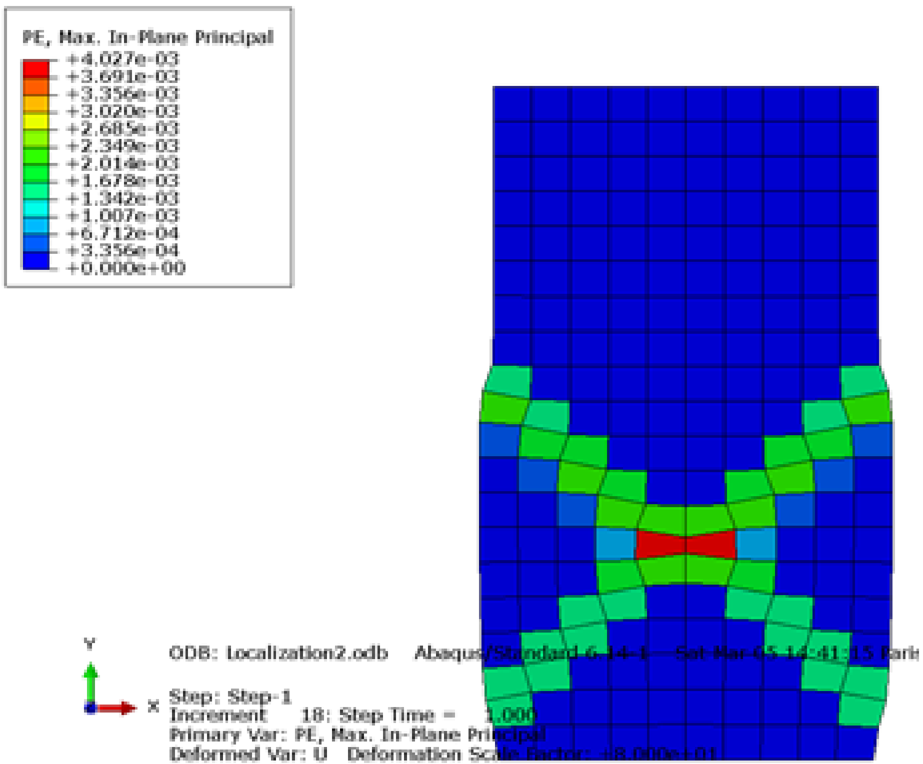

$d\varepsilon_p = \beta d\gamma_p > 0$
q
$d\gamma_p$
$\mu > 0$
$d\gamma_p$
$d\varepsilon_p = \beta d\gamma_p < 0$
1
$\mu = -\frac{dq}{dp'} < 0$
p'
p'_0

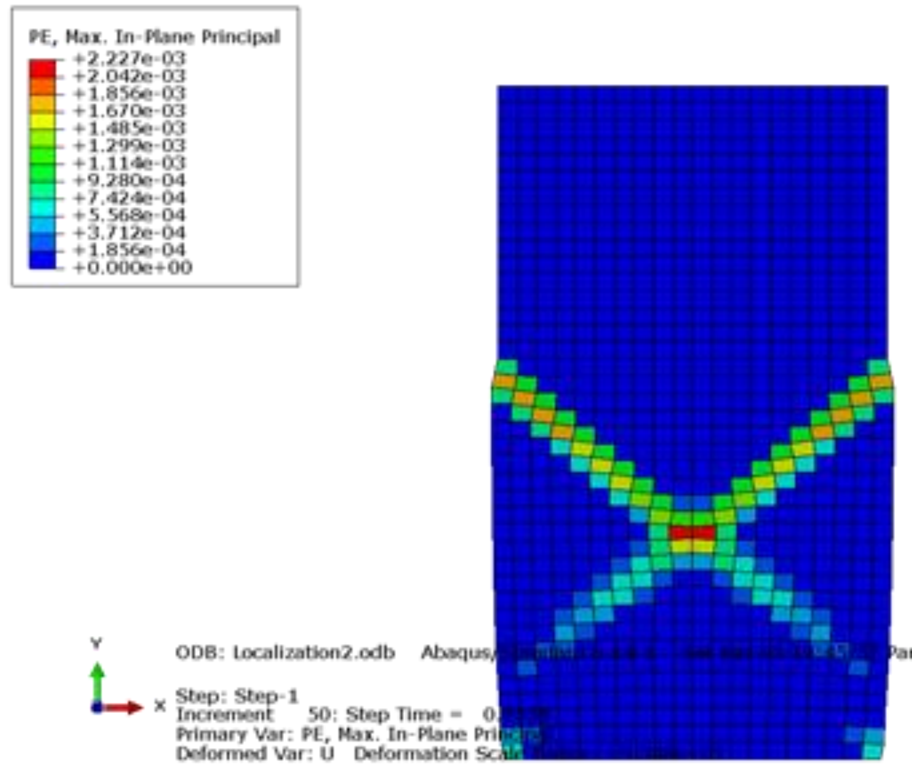

$\frac{h_{cr}}{G}$ for $h\downarrow$
left lower corner : compaction bands
right upper corner : dilation bands
in between : shear bands
β
μ

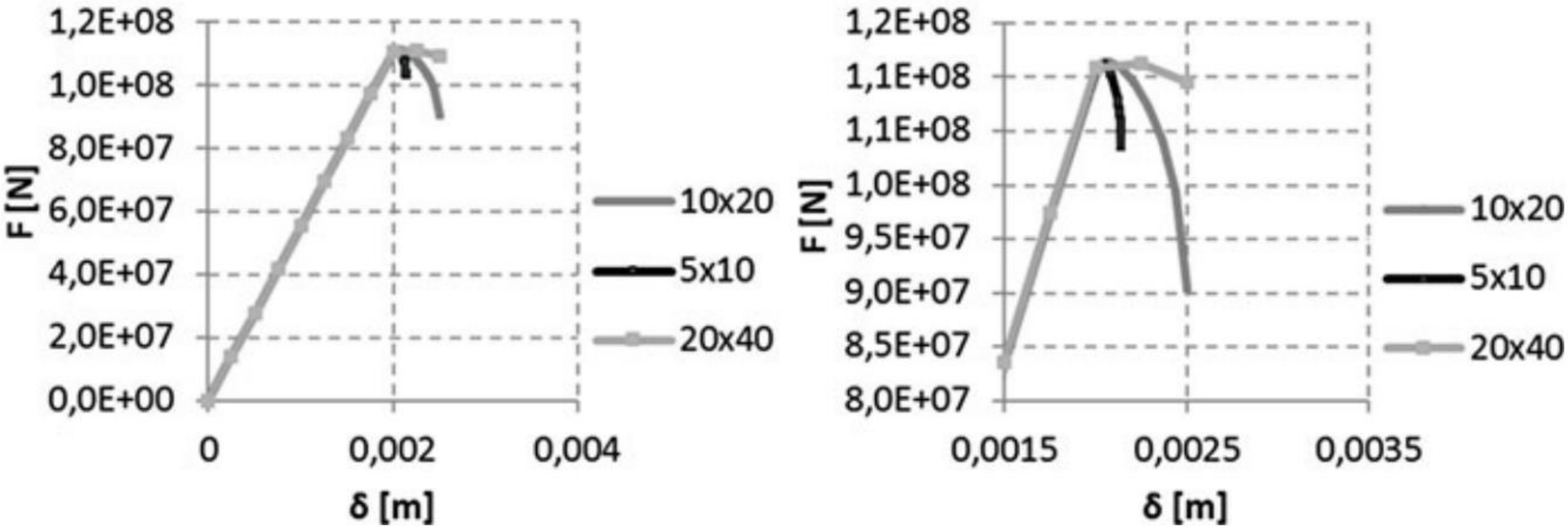

dilation band
dilatant shear band
shear band
contractant shear band
compaction band
n
g
n
g
n
g
n
g
n
g
q
p


PE, Max. In-Plane Principal
+4.027e-03
+3.691e-03
+3.356e-03
+3.020e-03
+2.685e-03
+2.349e-03
+2.014e-03
+1.678e-03
+1.342e-03
+1.007e-03
+6.712e-04
+3.356e-04
+0.000e+00
Y
X
ODB: Localization2.odb
Step: Step-1
Increment 18: Step Time = 1.000
Primary Var: PE, Max. In-Plane Principal
Deformed Var: U Deformation Scale Factor: +8.000e+01

Mesh dependency






1,2E+08
1,0E+08
8,0E+07
6,0E+07
4,0E+07
2,0E+07
0,0E+00
F [N]
0
0,002
0,004
δ [m]
10x20
5x10
20x40
1,2E+08
1,1E+08
1,1E+08
1,0E+08
9,5E+07
9,0E+07
8,5E+07
8,0E+07
F [N]
0,0015
0,0025
0,0035
δ [m]
10x20
5x10
20x40

Q11

Mathematical explanation

$$\rho\left(\frac{\lambda s}{2\pi}\right)^2 = sth > 0 \quad \Rightarrow \quad s = \frac{2\pi}{\lambda}\sqrt{\frac{sth}{\rho}}$$

$$\tilde{u}_i = U_i e^{st + i\frac{2\pi}{\lambda}n_p x_p}$$

The perturbation that propagates the fastest in the medium maximizes s and therefore minimizes λ.

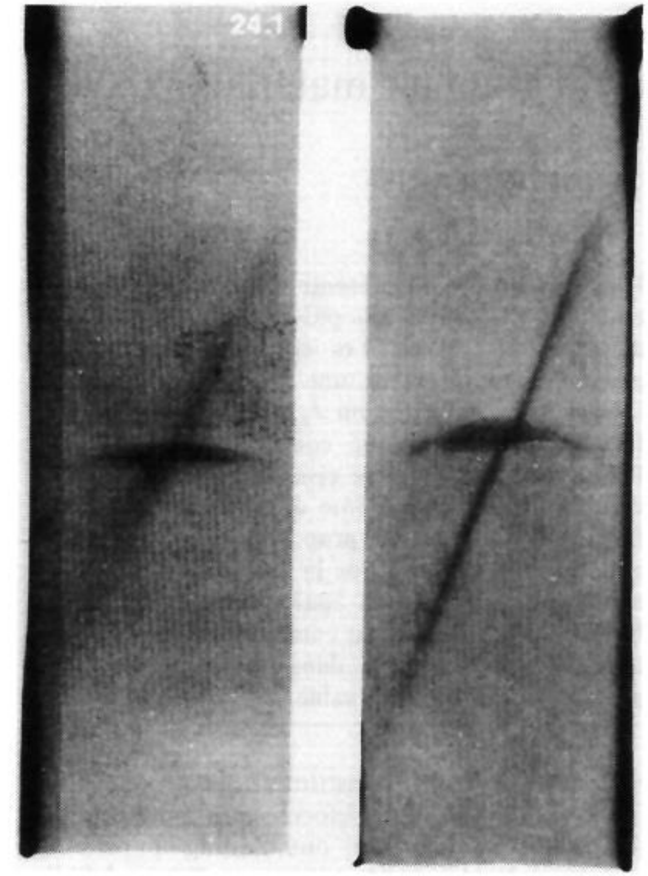
Localization happens on a mathematical plane ($\lambda \rightarrow 0$).

But this is not in accordance with experiments, which show that deformation bands have a **finite thickness**, controllable by the grain size (at least).

These experiments are very slow for the material to show any rate dependent sensitivity (Zheng et Zhao et al., 2013). So it seems not to be related to viscous effects, at least at 1st order.

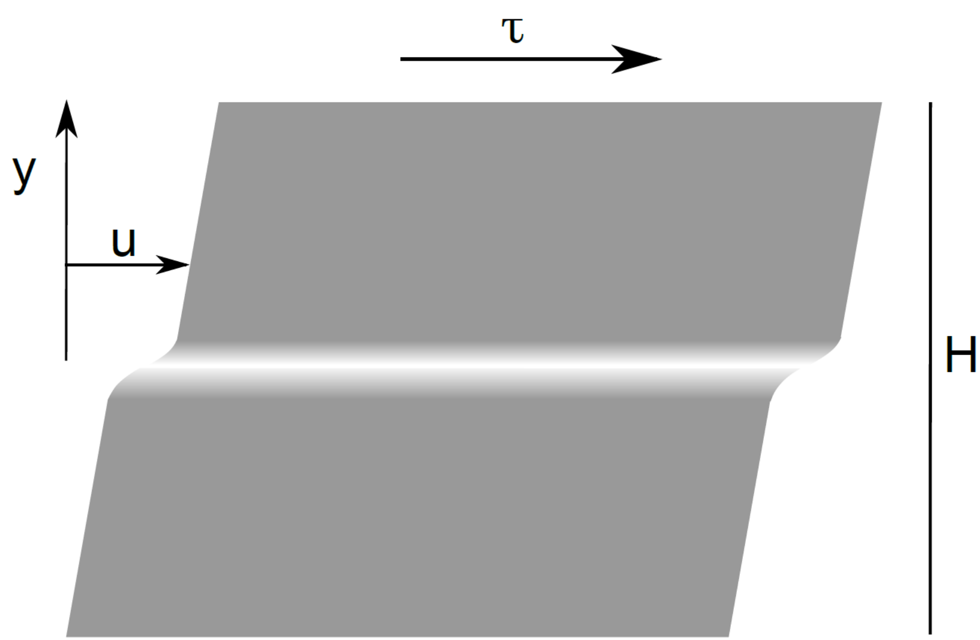
The reason seems to be the **absence of internal lengths in Cauchy medium**.

Higher order micromorphic continua, e.g. Cosserat (microstructure) and THM couplings are some approaches for inserting more physics into the problem leading to finite band thickness.



(Mühlhaus & Vardoulakis, 1987)

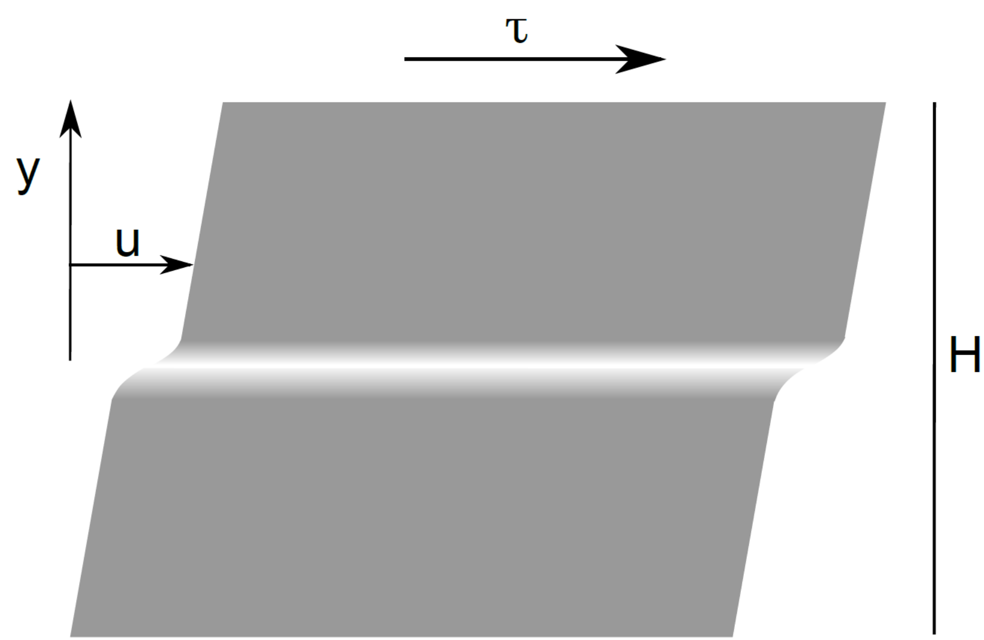
Exercise #1: Cauchy elasto-plastic layer



Exercise #1: Cauchy elasto-plastic layer

Elasto-plasticity:

$$F = \sigma_{12} - \tau_0 \leq 0$$



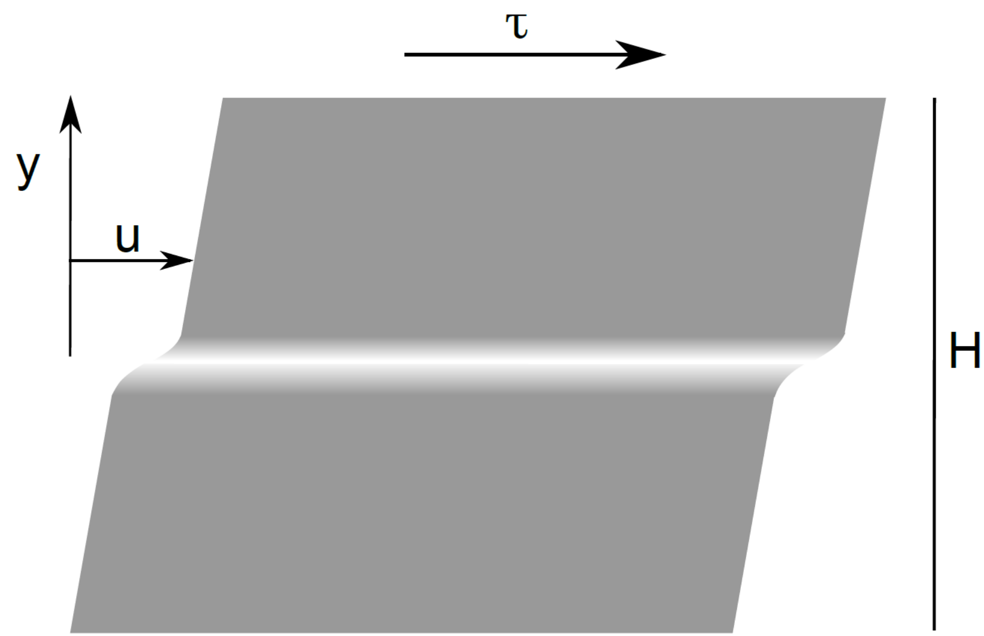
Exercise #1: Cauchy elasto-plastic layer

Elasto-plasticity:

$$F = \sigma_{12} - \tau_0 \leq 0$$

Small strains:

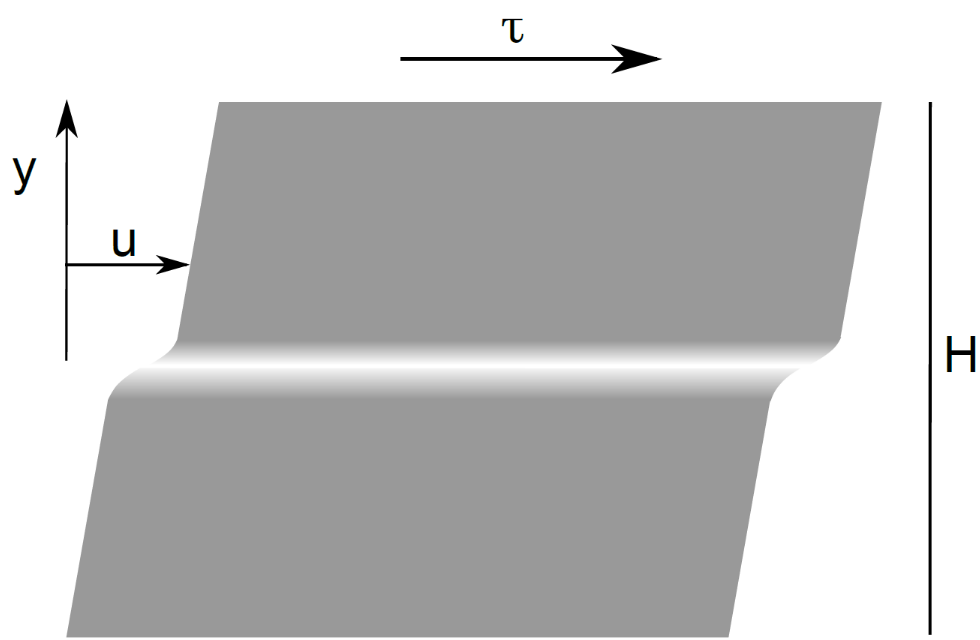
$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}$$



Exercise #1: Cauchy elasto-plastic layer

Elasto-plasticity:

$$F = \sigma_{12} - \tau_0 \leq 0$$



Small strains:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}$$

Linear elasticity:

$$\sigma_{ij} = K\varepsilon_{kk}^{el}\delta_{ij} + 2G\left(\varepsilon_{ij}^{el} - \frac{1}{3}\varepsilon_{kk}^{el}\delta_{ij}\right)$$

x_1, x_3 invariance and momentum balance:

$$\frac{\partial \sigma_{12}}{\partial x_2} = \rho \ddot{u}_1; \quad \frac{\partial \sigma_{22}}{\partial x_2} = \rho \ddot{u}_2$$

x_1, x_3 invariance and momentum balance:

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Steady-state:

$$\sigma_{12} = \sigma_{12}^{*} = \tau_0$$
$$\sigma_{22} = \sigma_{22}^{*} = \sigma_0$$

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$\Rightarrow$

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Steady-state:

$$\sigma_{12} = \sigma_{12}^* = \tau_0$$
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$$\Rightarrow \quad \frac{\partial \sigma_{12}^*}{\partial x_2} = 0; \qquad \frac{\partial \sigma_{22}^*}{\partial x_2} = 0$$

This state will be stable as long as any perturbations do not grow in time.

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$\Rightarrow$

$$\frac{\partial \sigma_{12}^*}{\partial x_2} = 0; \quad \frac{\partial \sigma_{22}^*}{\partial x_2} = 0$$

This state will be stable as long as any perturbations do not grow in time.

By perturbing the displacement fields: $u_i = u_i^* + \tilde{u}_i$

$$\frac{\partial \tilde{\sigma}_{12}}{\partial x_2} = \rho \ddot{\tilde{u}}_1; \quad \frac{\partial \tilde{\sigma}_{22}}{\partial x_2} = \rho \ddot{\tilde{u}}_2$$

Incremental law:

$$\tilde{\sigma}_{12} = 2G\tfrac{h}{1+h}\tilde{\varepsilon}_{12}$$

$$\tilde{\sigma}_{22} = M\tilde{\varepsilon}_{22}$$

where $M = K + \frac{4G}{3}$ is the p-wave elastic modulus,

$h = \frac{1}{G}\frac{d\tau_0}{d\alpha} > -1$ is the hardening modulus, with $\dot{\alpha} = \dot{\gamma}^{pl}_{(12)}$

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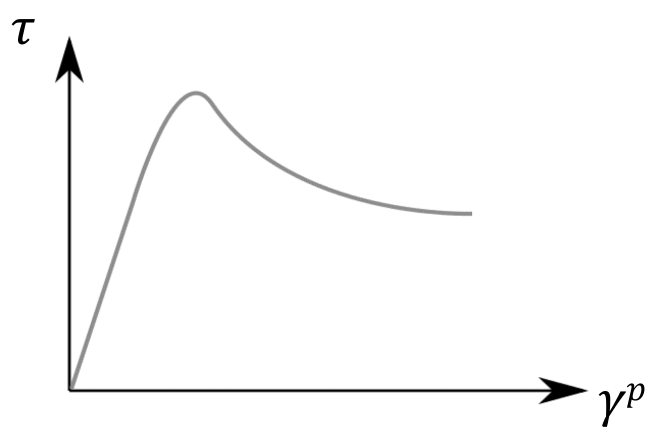

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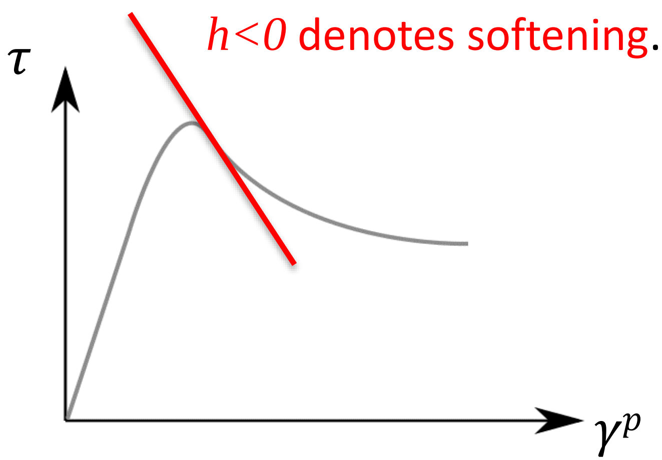

Boundary conditions:

The perturbations $\tilde{u}_i$ have to fulfill the boundary conditions:

$$\tilde{\sigma}_{12}\left(x_2=\pm\frac{H}{2}\right)=\tilde{\sigma}_{22}\left(x_2=\pm\frac{H}{2}\right)=0$$

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Direction of the shear band (imposed in this example)**:** $\{n_i\} = \{0, 1, 0\}$

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Direction of the shear band (imposed in this example)**:** $\{n_i\} = \{0, 1, 0\}$

General solution of $\frac{\partial \tilde{\sigma}_{12}}{\partial x_2} = \rho \ddot{\tilde{u}}_1; \quad \frac{\partial \tilde{\sigma}_{22}}{\partial x_2} = \rho \ddot{\tilde{u}}_2$ **:**

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General solution of $\frac{\partial \tilde{\sigma}_{12}}{\partial x_2} = \rho \ddot{\tilde{u}}_1; \quad \frac{\partial \tilde{\sigma}_{22}}{\partial x_2} = \rho \ddot{\tilde{u}}_2$ **:**

$$\tilde{u}_i = g_i e^{st + \mathrm{i} k n_j x_j} = g_i e^{st + \mathrm{i} k x}$$

Growth coefficient (Lyapunov exponent):

$$s = \mathrm{i}kv_p \quad \text{or}$$

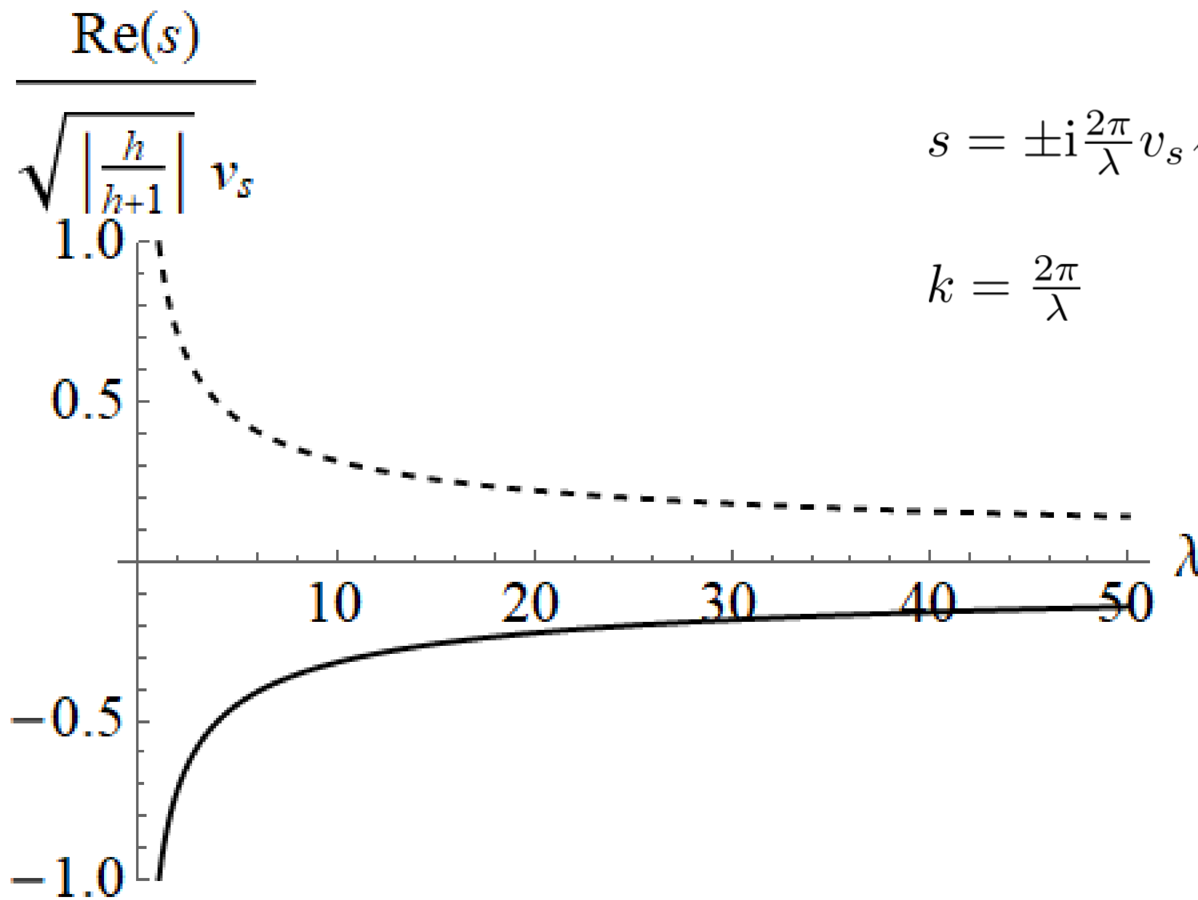
$$s = \pm\mathrm{i}kv_s\sqrt{\frac{h}{h+1}}$$

where $v_p = \sqrt{\frac{M}{\rho}}$ is the p-wave and $v_s = \sqrt{\frac{G}{\rho}}$ the shear velocity.

Q12

Instability of homogeneous (reference) deformation (=>localization):

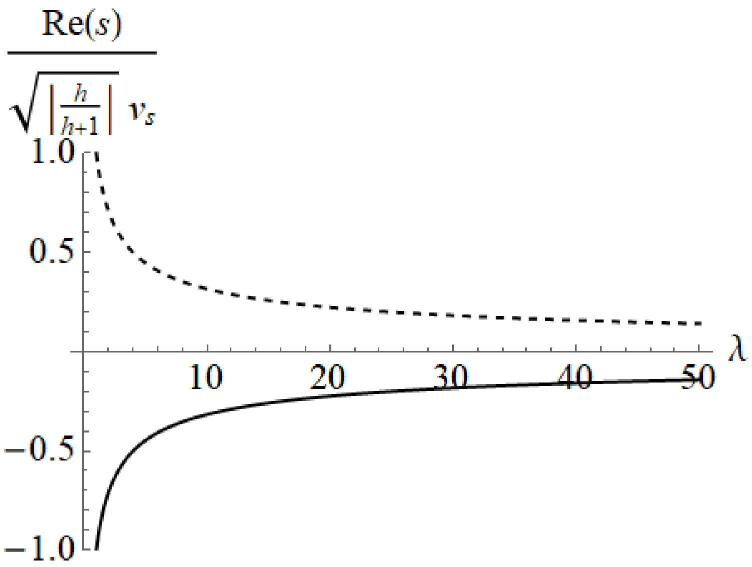
$$Re(s) > 0$$



$$s = \pm \mathrm{i}\frac{2\pi}{\lambda} v_s \sqrt{\frac{h}{h+1}}$$

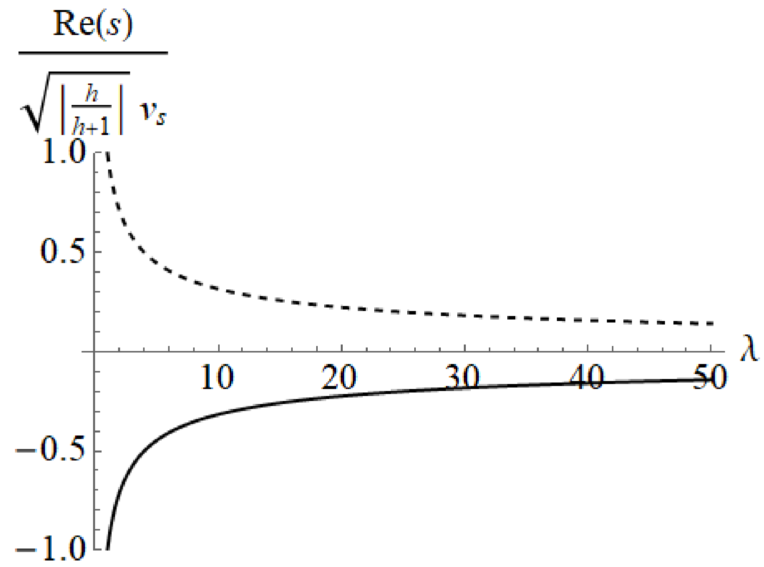
$$k = \frac{2\pi}{\lambda}$$

Summary of pathologies



1. Infinite rate of growth
2. Localization at zero wavelength/thickness (infinite wave number)

Summary of pathologies

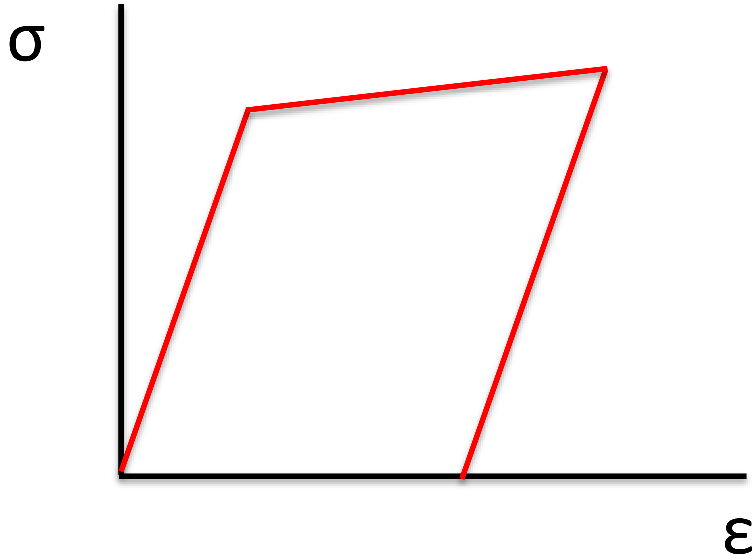


1. Infinite rate of growth
2. Localization at zero wavelength/thickness (infinite wave number)

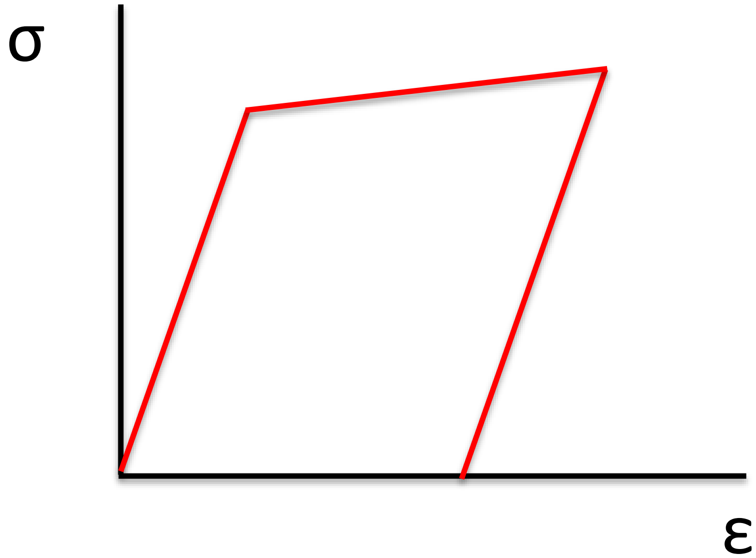
Lack of characteristic **time** and **length** scale

Constitutive behavior of solids

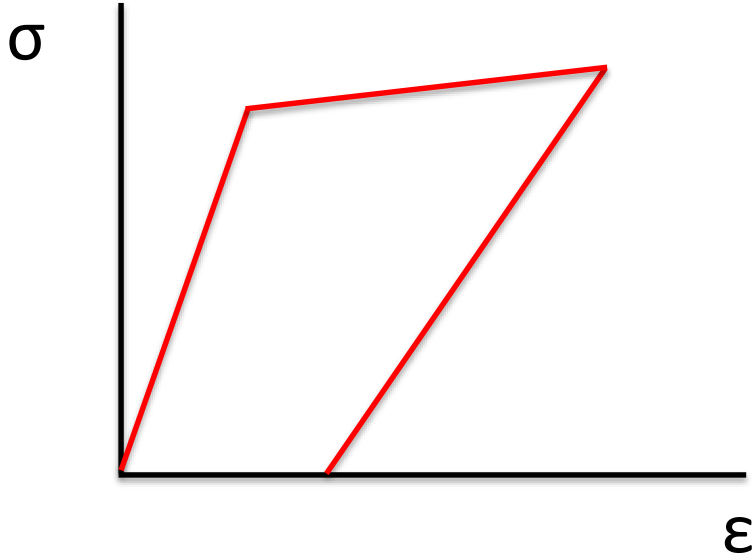
Q13



Elastoplasticity with hardening

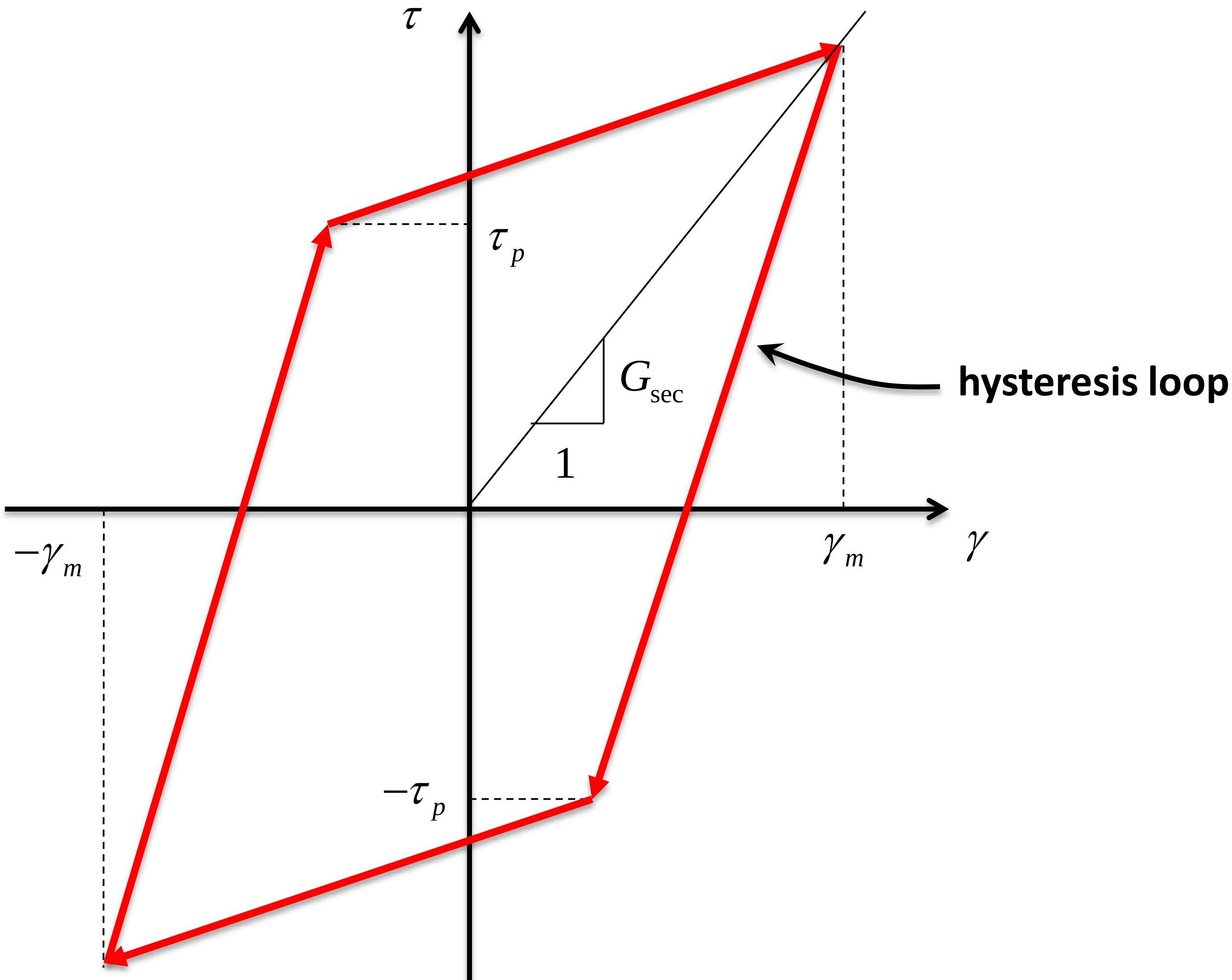


Elastoplasticity with
hardening



Elastoplasticity with
hardening and damage

Example: Cyclic loading



Q14

Viscous regularization (characteristic time)

Materials whose mechanical response depends on the rate of deformation are called viscous or rate-dependent:

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, \ldots\right)$$

Linearized form:

$$\tilde{\sigma}_{ij} = L_{ijkl}\tilde{\varepsilon}_{kl} + M_{ijkl}\dot{\tilde{\varepsilon}}_{kl}$$

Replacing into the balance equation:

$$\sigma_{ij,j} = \rho \ddot{u}_i,$$

yields:

$$L_{ijkl}\tilde{u}_{k,lj} + M_{ijkl}\dot{\tilde{u}}_{k,lj} = \rho \ddot{\tilde{u}}_i.$$

The above equation is linear and for deformation bands it takes solutions of the form $\tilde{u}_i = g_i e^{\mathrm{i}kn_i x_i + st}$.

Finally:

$$\left[n_j L_{ijkl} n_l + \boxed{n_j M_{ijkl} n_l s} + \rho \left(\frac{s}{k}\right)^2 \delta_{ik}\right] g_k = 0$$

or

$$\left[\Gamma_{ik} + \Delta_{ik} s + \rho \left(\frac{\lambda s}{2\pi}\right)^2 \delta_{ik}\right] g_k = 0$$

Scaling

Let

$$\tau = \frac{t}{T}, \quad \chi_k = \frac{x_k}{L},$$

where T is a characteristic time and L a characteristic length.

Then we obtain:

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s \hat{k} T}\right)^2 \hat{s}^2 \delta_{ik}\right] g_k = 0,$$

where v_s is the shear-wave velocity, $v_s = \sqrt{\frac{G}{\rho}}$, $\hat{s} = sT$ and $\hat{k} = kL$.

Case #1: Negligible inertia

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s \hat{k} T}\right)^2 \hat{s}^2 \delta_{ik}\right] g_k = 0$$

Let $\frac{\Gamma_{ik}}{G}$ and $\frac{\Delta_{ik}}{GT}$ are terms of $O(1)$ and $\frac{L^2}{v_s^2 \hat{k}^2 T^2}$ of $O(\varepsilon)$, $\varepsilon \ll 1$.

- $\frac{\Delta_{ik}}{GT_{visc}} = c_{ik} \approx O(1)$ leads to:

$$T_{visc} = c_{ik} \frac{\Delta_{ik}}{G}.$$

- $\frac{L^2}{v_s^2 \hat{k}^2 T_{visc}^2} \ll 1$ yields:

$$T_{visc} \gg \frac{L}{v_s \hat{k}} \Rightarrow c_{ik} \frac{\Delta_{ik}}{G} \gg \frac{L\hat{\lambda}}{v_s 2\pi} \Rightarrow$$

$$\hat{\lambda} \ll 2\pi v_s \frac{c_{ik}\Delta_{ik}}{GL} \equiv \hat{\lambda}^*.$$

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s \hat{k} T}\right)^2 \hat{s}^2 \delta_{ik}\right] g_k = 0$$

So, when $\lambda \ll \lambda^* = 2\pi v_s T_{visc}$ inertia terms can be dropped:

$$\left(\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{G}\frac{\hat{s}}{T_{visc}} + \varepsilon \hat{s}^2 \delta_{ik}\right) g_k = 0 \Rightarrow$$

$$\boxed{\left(\frac{\Gamma_{ik}}{G} + c_{ik} s\right) g_k = 0}$$

Assuming strain localization in an isotropic rock with:

$G \approx 30\text{GPa}$, $c_{ij}\Delta_{ij} = \eta \approx 20\text{MPas}$ and $v_s \approx 2000\text{m/s}$

then $\lambda^* \simeq 8m$, which is much larger than the localization thickness of a deformation band (some millimeters or even smaller).

Therefore, for typical applications viscosity effects dominate over inertial ones. In other words:

License to kill inertia!

(for typical localization problems)

Case #2: Negligible viscosity

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s \hat{k} T}\right)^2 \hat{s}^2 \delta_{ik}\right] g_k = 0$$

Suppose $\frac{\Gamma_{ik}}{G}$ and $\frac{L^2}{v_s^2 \hat{k}^2 T^2}$ are terms of $O(1)$ and $\frac{\Delta_{ik}}{GT}$ of $O(\varepsilon)$.

- $\frac{L^2}{v_s^2 \hat{k}^2 T^2} \approx O(1)$ results:

$$T_{iner} = \frac{L}{v_s \hat{k}} = \frac{\hat{\lambda} L}{2\pi v_s}.$$

- $\frac{\Delta_{ik}}{GT} \ll 1$ yields:

$$c_{ik}\frac{\Delta_{ik}}{G} \ll T_{iner} = \frac{\hat{\lambda} L}{2\pi v_s} \Rightarrow$$

$$\hat{\lambda} \gg \hat{\lambda}^* = 2\pi v_s \frac{c_{ik}\Delta_{ik}}{GL}.$$

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s \hat{k} T}\right)^2 \hat{s}^2 \delta_{ik}\right] g_k = 0$$

So for very large wave lengths $\lambda \gg \lambda^*$ viscosity terms can be dropped:

$$\left(\frac{\Gamma_{ik}}{G} + s^2 \delta_{ik}\right) g_k = 0$$

Case #3: Negligible rate-independency

$$\left[\frac{\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \left(\frac{L}{v_s\hat{k}T}\right)^2 \hat{s}^2\delta_{ik}\right] g_k = 0$$

We assume a new time-scale, such that $\tau_{v\&i} = \varepsilon^{-a}\tau$. This leads to:

$$\left[\frac{\varepsilon^a\Gamma_{ik}}{G} + \frac{\Delta_{ik}}{GT}\hat{s} + \varepsilon^{-a}\left(\frac{L}{v_s\hat{k}T}\right)^2 \hat{s}^2\delta_{ik}\right] g_k = 0$$

Assuming $\varepsilon^{-a}\left(\frac{L}{v_s\hat{k}T}\right)^2$ and $\frac{\Delta_{ik}}{GT}$ to be terms of $O(1)$ and $\varepsilon^a\frac{\Gamma_{ik}}{G}$ of $O(\varepsilon)$ with $\varepsilon \ll 1$ we obtain that $a = 1$, $T = T_{visc}$ and $\varepsilon = \frac{L^2}{v_s^2\hat{k}^2T^2} = \frac{T_{inertia}^2}{T_{visc}^2}$.
Therefore we get:

$$\left(\frac{\varepsilon\Gamma_{ik}}{G} + c_{ik}\hat{s} + \hat{s}^2\delta_{ik}\right) g_k = 0 \Rightarrow \boxed{\left(c_{ik}\hat{s} + \hat{s}^2\delta_{ik}\right) g_k = 0}$$

and

$$T_{v\&i} = \varepsilon T_{visc} = \frac{T_{iner}^2}{T_{visc}} < T_{inner} < T_{visc}$$

Summarizing:

Depending on the material parameters and the characteristic time of the phenomenon we study we can have:

- $T = T_{iner} \ll T_{visc}$ $(\lambda \ll \lambda^*)$ inertia terms can be dropped:

$$\left(\frac{\Gamma_{ik}}{G} + c_{ik}s\right) g_k = 0$$

Localization thinkness depends on the perturbation (no wave-length selection) and the rate of growth s is finite

- $T = T_{visc} \ll T_{iner}$ $(\lambda \gg \lambda^*)$ viscosity terms can be dropped:

$$\left(\frac{\Gamma_{ik}}{G} + s^2\delta_{ik}\right) g_k = 0$$

Localization thinkness is zero and the rate of growth s is infinite

- $T = T_{v\&i} \ll T_{iner} \ll T_{visc}$ rate-independent terms can be dropped:

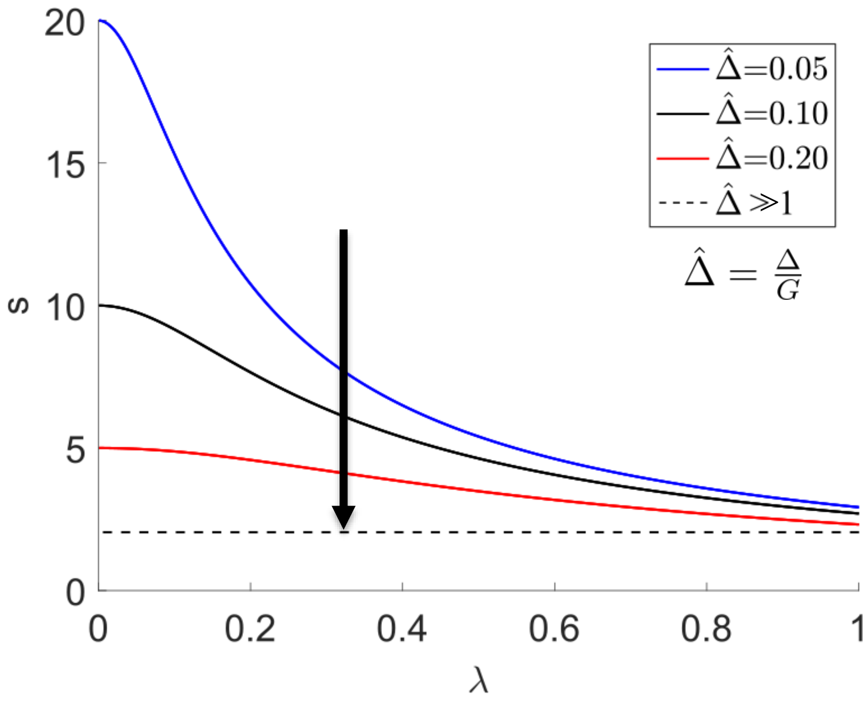
$$\left(c_{ik}\hat{s} + \hat{s}^2\delta_{ik}\right) g_k = 0$$

If the material is not rate-softening no localization happens

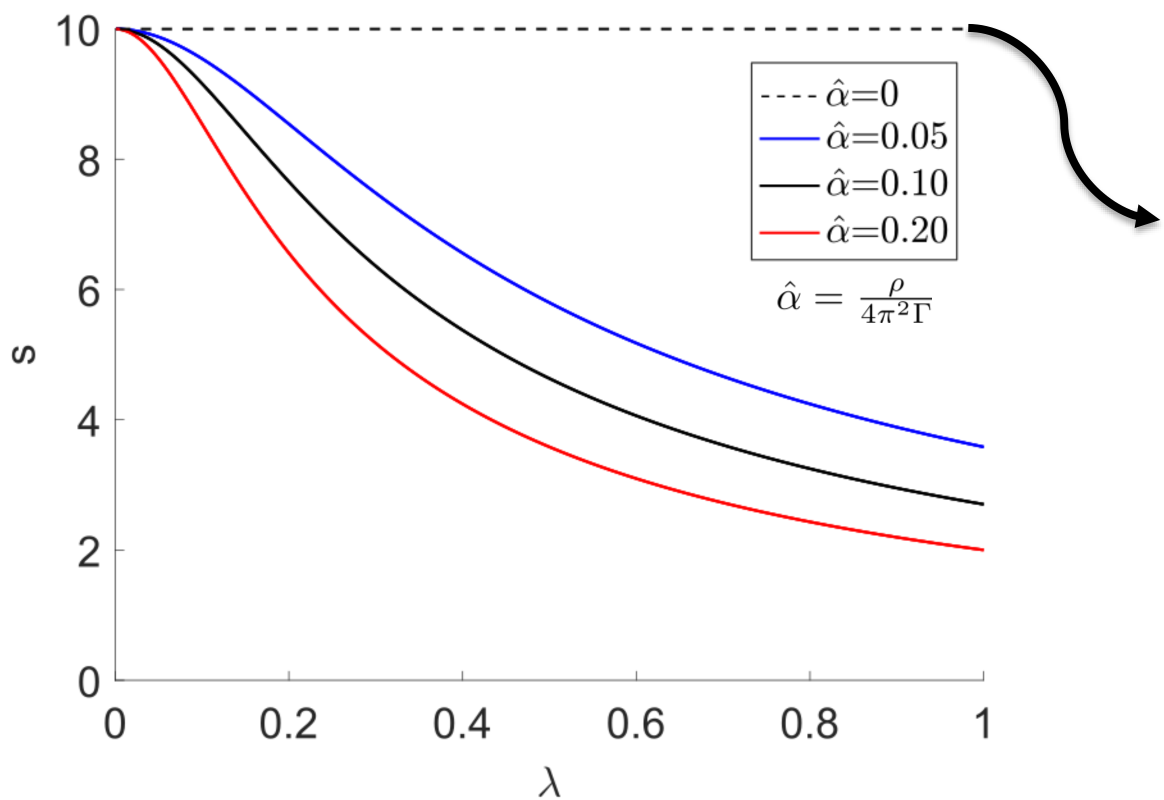
1D example

$$\left[\Gamma_{ik} + \Delta_{ik}s + \rho\left(\frac{s}{k}\right)^2\delta_{ik}\right]g_k = 0 \rightarrow \quad \Gamma + \Delta s + \rho\left(\frac{s}{k}\right)^2 = 0$$

$$\frac{\Gamma}{G} + \frac{\Delta}{G}s + \left(\frac{\lambda}{2\pi v_s}\right)^2 s^2 = 0$$



- Perturbation growing fastest has $\lambda=0$
- s is finite

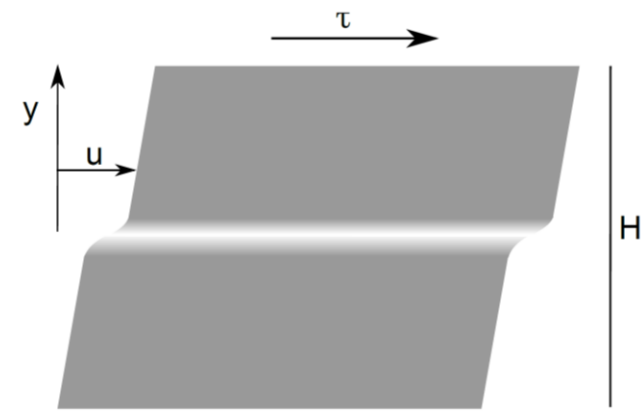


All perturbations propagate with the same rate:
No wave-length selection

$$\frac{\Gamma}{G} + \frac{\Delta}{G}s + \left(\frac{\lambda}{2\pi v_s}\right)^2 s^2 = 0$$

See also:
Needleman, 1988
Wang et al., 1997

Exercise #2: Perzyna layer



Elasto-visco-plasticity:

$$F = \sigma_{12} - \tau_0$$

Deformation is split in elastic and viscoplastic parts:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij}^{vpl}$$

According to Perzyna (1966):

$$\dot{\varepsilon}_{ij}^{vp} = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}} = \frac{F}{\eta f_0}\frac{\partial F}{\partial \sigma_{ij}},$$

where η is the viscosity and $f_0 = \tau_0$.

From the definition of the plastic multiplier $\dot{\lambda}$:

$$\dot{F} = \eta f_0 \ddot{\lambda} \Rightarrow \dot{\sigma}_{12} = 2G\frac{h}{1+h}\dot{\varepsilon}_{12} + \frac{\eta f_0}{1+h}\ddot{\lambda}$$

And finally:

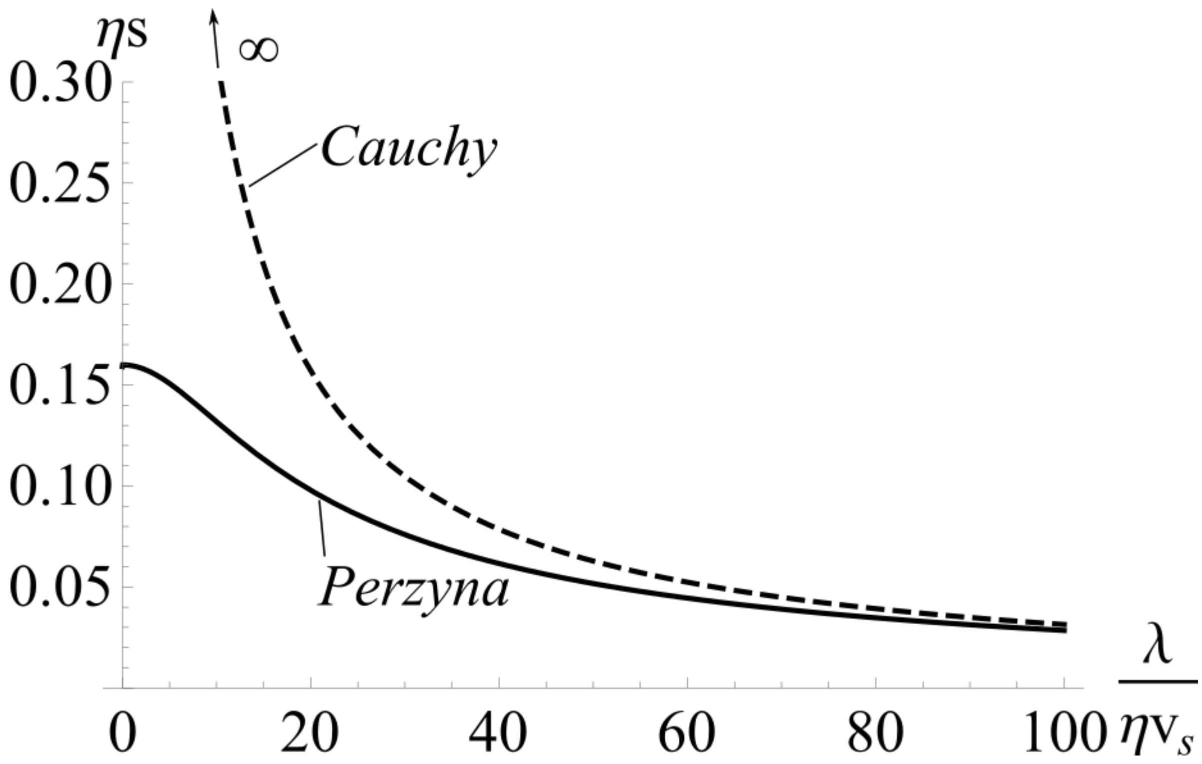
$$\tilde{\sigma}_{12} = 2G\frac{h}{1+h}\tilde{\varepsilon}_{12} + 2\frac{\eta f_0}{(1+h)^2}\dot{\tilde{\varepsilon}}_{12} - 2\frac{(\eta f_0)^2}{G(1+h)^3}\ddot{\tilde{\varepsilon}}_{12} + 2\frac{(\eta f_0)^3}{G^2(1+h)^4}\dddot{\tilde{\varepsilon}}_{12} - \dots$$

And finally:

$$\tilde{\sigma}_{12} = 2G\frac{h}{1+h}\tilde{\varepsilon}_{12} + 2\frac{\eta f_0}{(1+h)^2}\dot{\tilde{\varepsilon}}_{12} - 2\frac{(\eta f_0)^2}{G(1+h)^3}\ddot{\tilde{\varepsilon}}_{12} + 2\frac{(\eta f_0)^3}{G^2(1+h)^4}\dddot{\tilde{\varepsilon}}_{12} - \dots$$

And finally:

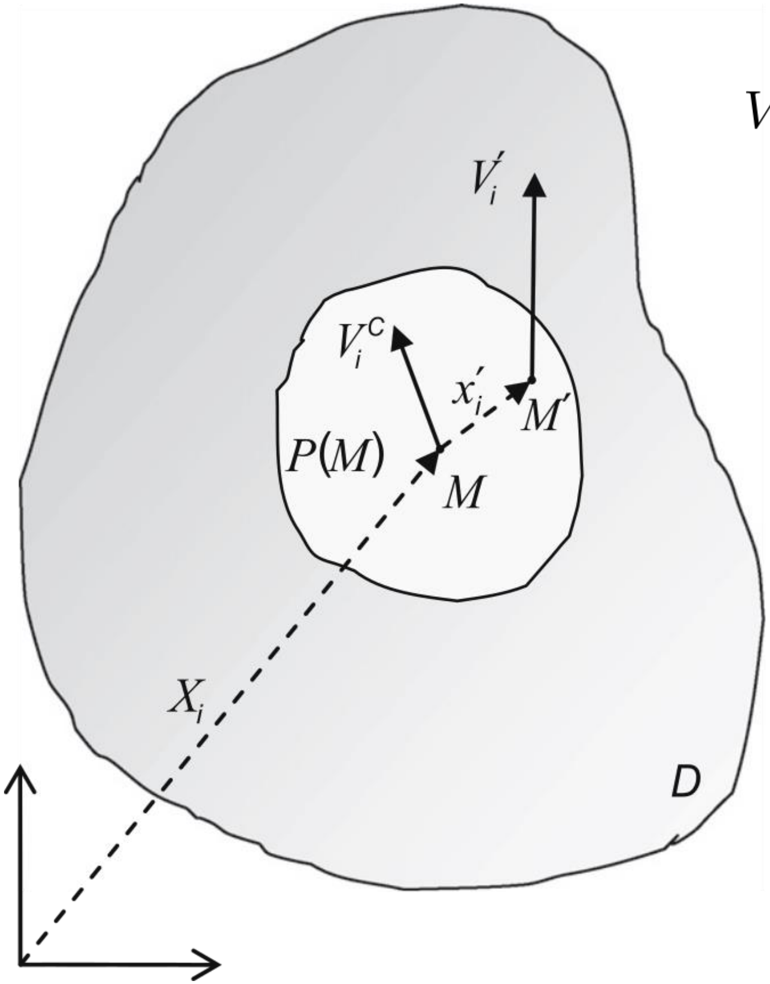
$$\tilde{\sigma}_{12} = 2G\frac{h}{1+h}\tilde{\varepsilon}_{12} + 2\frac{\eta f_0}{(1+h)^2}\dot{\tilde{\varepsilon}}_{12} - 2\frac{(\eta f_0)^2}{G(1+h)^3}\ddot{\tilde{\varepsilon}}_{12} + 2\frac{(\eta f_0)^3}{G^2(1+h)^4}\dddot{\tilde{\varepsilon}}_{12} - \dots$$



For $\hat{\lambda} \gg \hat{\lambda}^* \approx 20$ inertia is dominant.

Regularization with micromorphic continua (characteristic length)

Ansatz



$$V_i' = V_i + \chi_{ij} x_j' + \chi_{ijk} x_j' x_k' + \chi_{ijkl} x_j' x_k' x_l' + \ldots$$

(Germain, 1973, Mindlin, 1964 Eringen, 1999, …)

Strong form of micromorphic continua

$$
\begin{aligned}
\tau_{ij,j} + f_i = 0, &\qquad t_i = \tau_{ij} n_j \\
\nu_{ijk,k} + s_{ij} + \psi_{ij} = 0, &\qquad \mu_{ij} = \nu_{ijk} n_k \\
\nu_{ijkl,l} + s_{ijk} + \psi_{ijk} = 0, &\qquad \mu_{ijk} = \nu_{ijkl} n_l \\
\ldots &
\end{aligned}
$$

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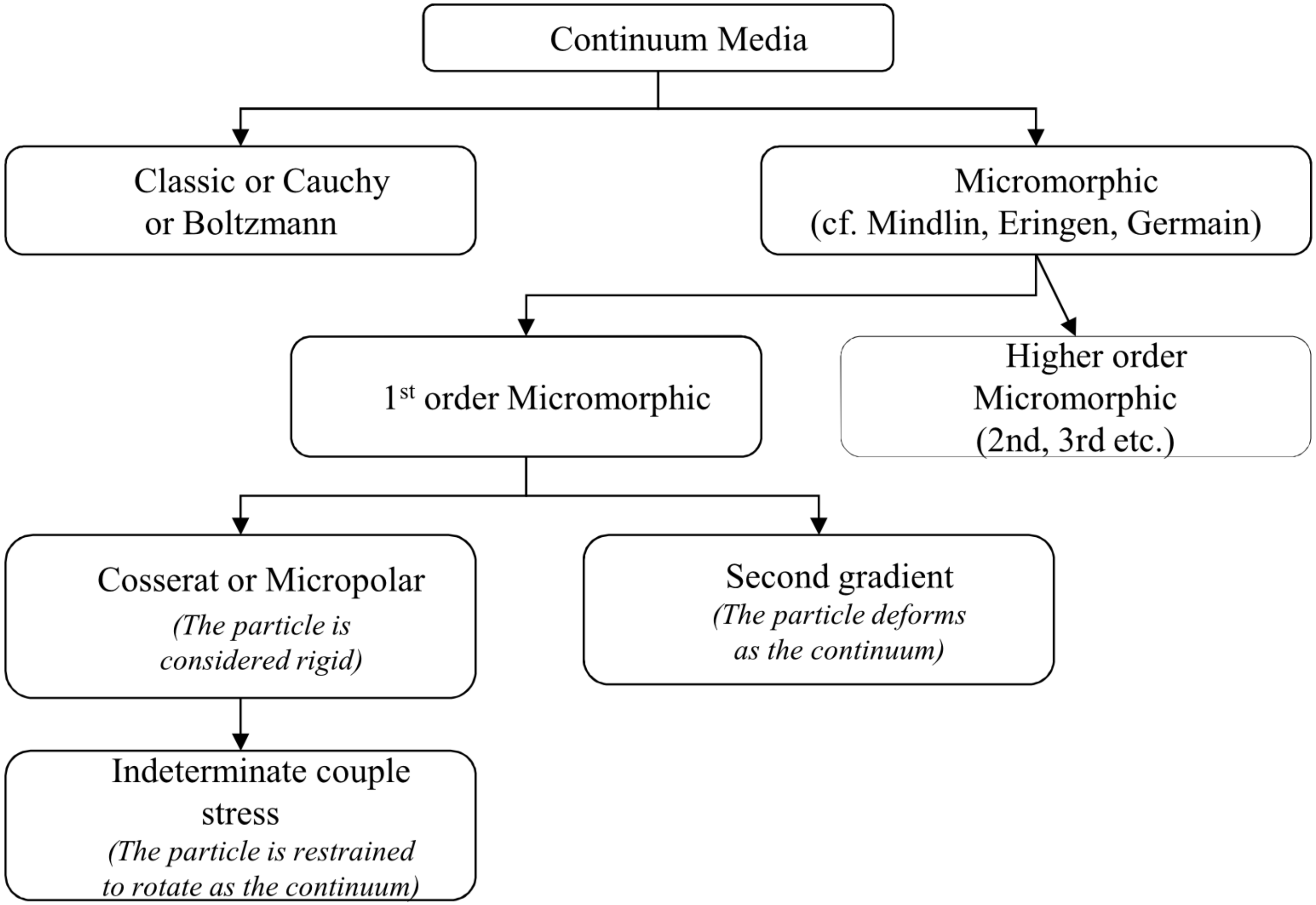
$$V_i' = V_i + \chi_{ij} x_j' + \chi_{ijk} x_j' x_k' + \chi_{ijkl} x_j' x_k' x_l' + \dots$$

Classification



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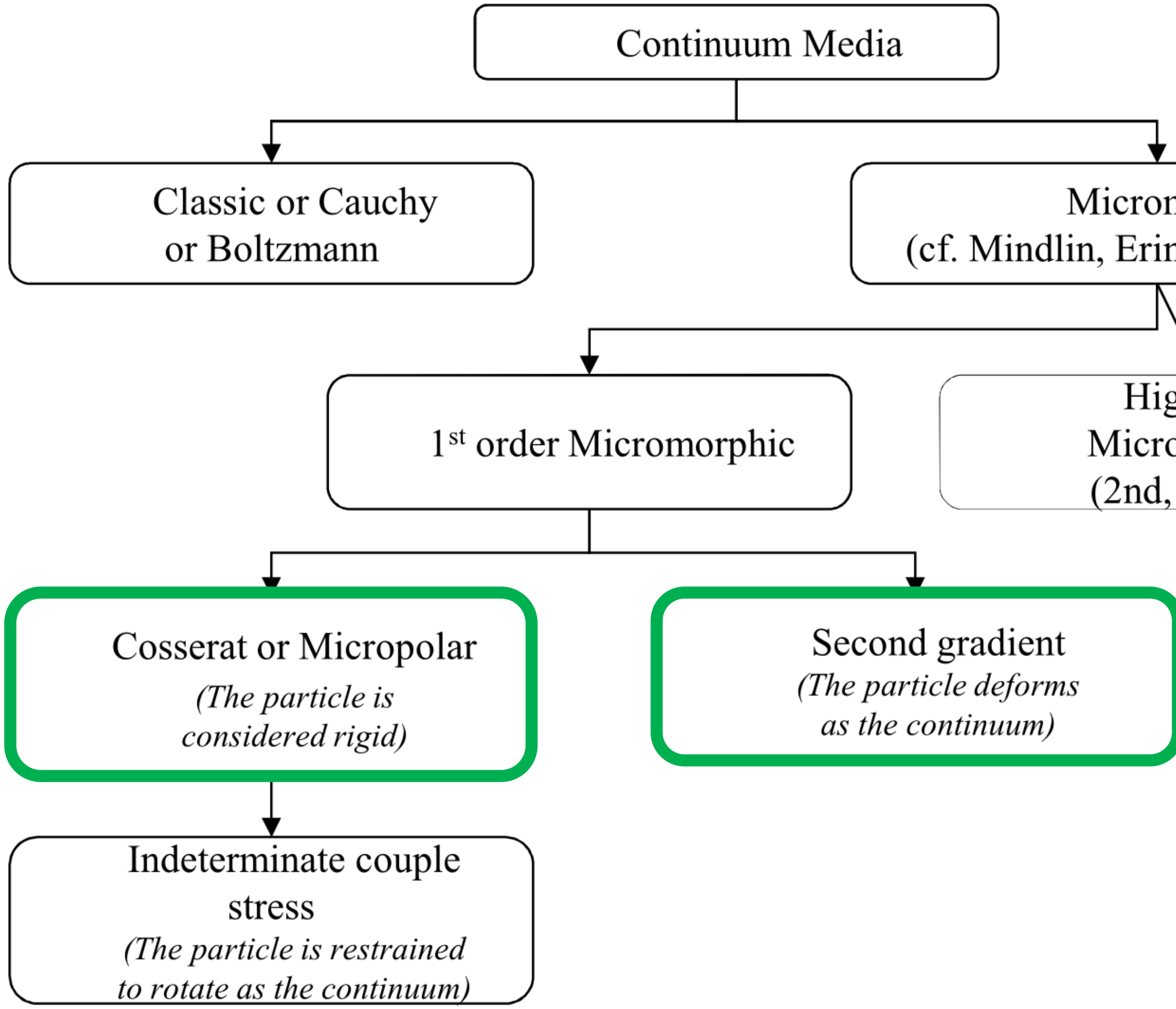

Classification

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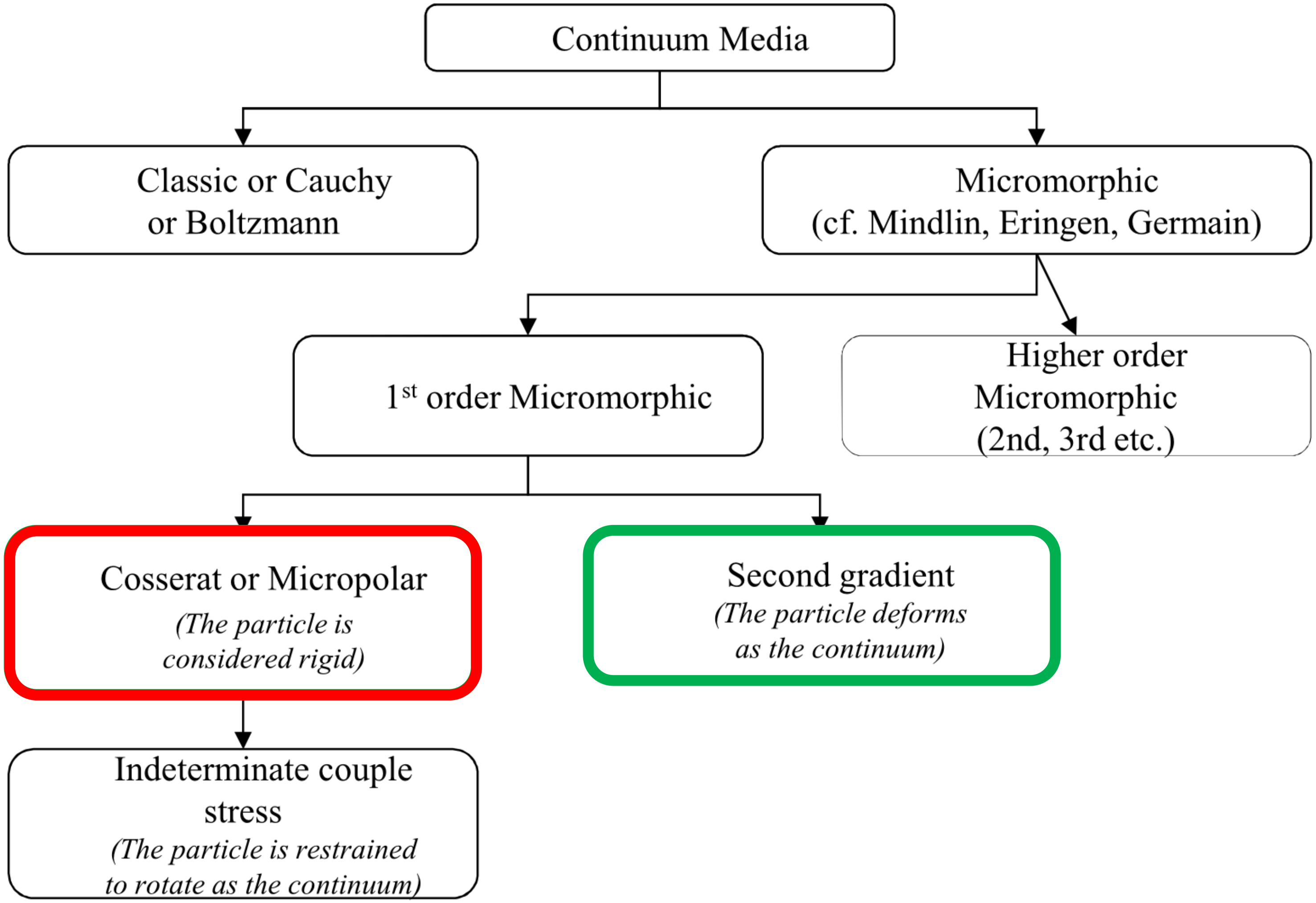

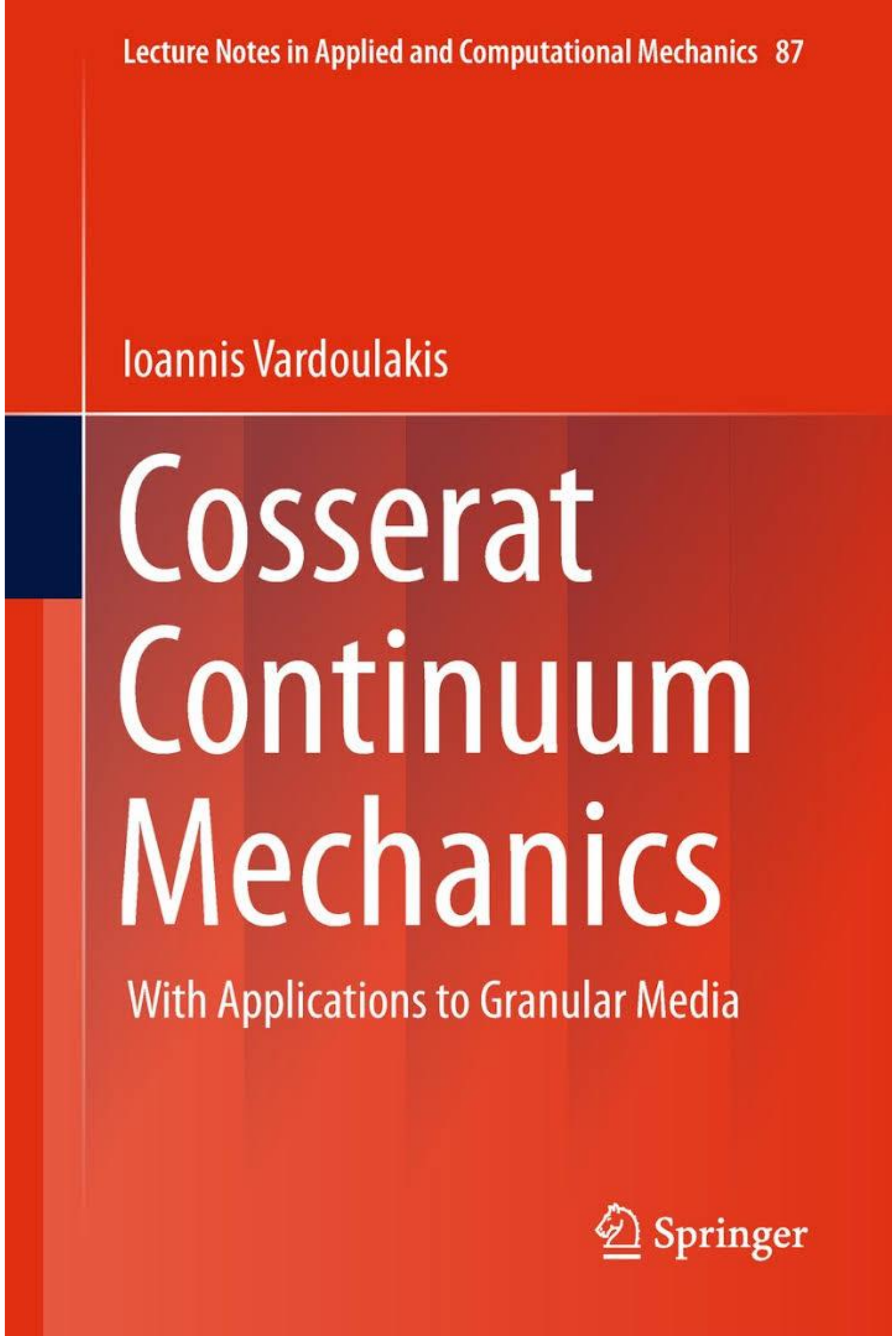
Lecture Notes in Applied and Computational Mechanics 87
Ioannis Vardoulakis
Cosserat Continuum Mechanics
With Applications to Granular Media
Springer

Momentum balance

$$\tau_{ij,j} + f_i = \rho \ddot{u}_i, \qquad t_i = \tau_{ij} n_j$$
$$m_{ij,j} - \epsilon_{ijk}\tau_{jk} + \psi_i = I\ddot{\omega}_i^c, \qquad \mu_i = m_{ij} n_j$$

τ_{ij} is the Cosserat stress tensor (non-symmetric)

m_{ij} is the Cosserat moment (couple stress) tensor

u_i and ω_i^c are the Cosserat displacements and rotations

t_i and μ_i denote boundary tractions

I is the microinertia

ρ is the density

Constitutive law, perturbation and linearization

Constitutive law: $\tau_{ij} = \tau_{ij}(\gamma_{ij}, \kappa_{ij})$ and $m_{ij} = m_{ij}(\gamma_{ij}, \kappa_{ij})$

$$\begin{aligned}\gamma_{ij} =& u_{i,j} + \epsilon_{ijk}\omega_k^c \\ \kappa_{ij} =& \omega_{i,j}^c\end{aligned}$$

We perturb the kinematic fields u_i and ω_i as follows:

$$\begin{aligned}\tilde{u}_i =& u_i - u_i^* = U_i e^{st+k_j n_j} \\ \tilde{\omega}_i^c =& \omega_i^c - \omega_i^{c*} = \Omega_i e^{st+k_j n_j}\end{aligned}$$

Linearization of the constitutive law yields:

$$\begin{aligned}\tilde{\tau}_{ij} =& C_{ijkl}^{TT}\tilde{\gamma}_{kl} + C_{ijkl}^{TM}\tilde{\kappa}_{kl} \\ \tilde{m}_{ij} =& C_{ijkl}^{MT}\tilde{\gamma}_{kl} + C_{ijkl}^{MM}\tilde{\kappa}_{kl}\end{aligned}$$

Eigenvalue problem

Replacing:

$$\begin{bmatrix} \Gamma_{ik} + \rho \left(\frac{s}{k}\right)^2 \delta_{ik} & \Delta_{ik} \\ \Xi_{ik} & \Pi_{ik} + I \left(\frac{s}{k}\right)^2 \delta_{ik} \end{bmatrix} \begin{bmatrix} U_k \\ \Omega_k \end{bmatrix} = 0,$$

where

$$\Gamma_{ik} = n_j C^{TT}_{ijkl} n_l$$

$$\Delta_{ik} = -\mathrm{i}\frac{1}{k} n_j e_{qlk} C^{TT}_{ijql} + n_j C^{TM}_{ijkl} n_l$$

$$\Xi_{ik} = n_j C^{MT}_{ijkl} n_l + \mathrm{i}\frac{1}{k} e_{ijr} C^{TT}_{jrkq} n_q$$

$$\Pi_{ik} = n_j C^{MM}_{ijkl} n_l - \mathrm{i}\frac{1}{k} e_{rnk} C^{MT}_{ilrn} n_l + \frac{1}{k^2} e_{ilr} C^{TT}_{lrnq} e_{nqk} + \mathrm{i}\frac{1}{k} e_{ilr} C^{TM}_{lrkq} n_q$$

Condition for strain localization

$$\mathrm{Det}\left(\begin{bmatrix}\Gamma_{ik}-\rho c^2\delta_{ik} & \Delta_{ik}\\ \Xi_{ik} & \Pi_{ik}-Ic^2\delta_{ik}\end{bmatrix}\right)=0$$

(Steinmann & Willam 1991, Iordache & Willam 1998, Rattez et al. 2018)

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(Steinmann & Willam 1991, Iordache & Willam 1998, Rattez et al. 2018)

Removing Cosserat effects:

$$\mathrm{Det}\left(\Gamma_{ik}-\rho c^2\delta_{ik}\right)=0$$

Application: Mühlhaus-Vardoulakis plasticity model

Strain hardening elasto-plasticity for 3D Cosserat continuum:

$$F = \tau + \mu\sigma, \qquad Q = \tau + \beta\sigma$$

$$\sigma = \sigma_{ii}/3; \qquad \tau = \sqrt{h_1 s_{ij} s_{ij} + h_2 s_{ij} s_{ji} + (h_3 m_{ij} m_{ij} + h_3 m_{ij} m_{ji})/R^2} \qquad \{h_i\} = \{2/3, -1/6, 2/3, -1/6\}$$

$$\{g_i\} = \{8/5, 2/5, 8/5, 2/5\}$$

$$\dot{\varepsilon}^p = \dot{\varepsilon}^p_{kk}; \qquad \dot{\gamma}^p = \sqrt{g_1 \dot{e}^p_{ij} \dot{e}^p_{ij} + g_2 \dot{e}^p_{ij} \dot{e}^p_{ji} + (g_3 \dot{\kappa}^p_{ij} \dot{\kappa}^p_{ij} + g_4 \dot{\kappa}^p_{ij} \dot{\kappa}^p_{ji}) R^2}$$

Mühlhaus, Vardoulakis (1988)
Rattez et al. (2018)

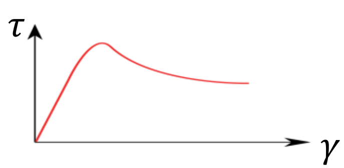
s_{ij} and e_{ij} are the deviatoric parts of the stress and strain tensors respectively,

$$\dot{\varepsilon} = \dot{\varepsilon}^{el} + \dot{\varepsilon}^p = \frac{1}{K}\dot{\sigma} + \dot{\varepsilon}^p, \qquad \dot{\varepsilon}^p = \beta\dot{\gamma}^p$$

$$\dot{\gamma} = \dot{\gamma}^{el} + \dot{\gamma}^p = \frac{1}{G}\dot{\tau} + \dot{\gamma}^p, \qquad \dot{\gamma}^p = \frac{1}{H}\left(\dot{\tau} + \mu\dot{\sigma}\right)$$

where $H = H\left(\gamma^p\right) = h\left(\sigma + p\right)$ is the plastic hardening modulus ($h = d\mu / d\gamma^p$) which is related to the tangent modulus $H_{tan} = \dfrac{H}{1 + H/G}$

and it is either positive (hardening) or negative (softening)



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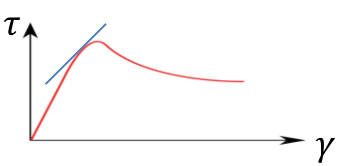

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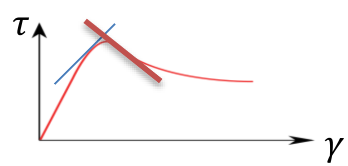
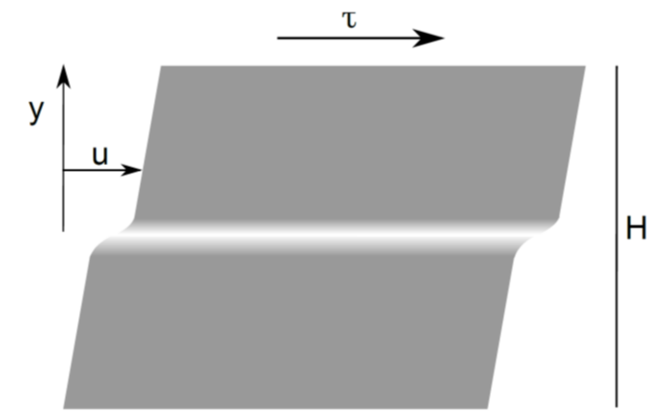


Exercise #3: Cosserat layer



Yield surface: $F = \tau_{(12)} - \tau_0 \leq 0$

Strains and curvatures split:

$$\dot{\gamma}_{ij} = \dot{\gamma}_{ij}^{el} + \dot{\gamma}_{ij}^{pl}$$
$$\dot{\kappa}_{ij} = \dot{\kappa}_{ij}^{el} + \dot{\kappa}_{ij}^{pl}$$

Incremental constitutive law:

$$\tilde{\tau}_{(12)} = 2G\frac{h}{1+h}\tilde{\gamma}_{(12)}$$
$$\tilde{\tau}_{[12]} = 2G_c\tilde{\gamma}_{[12]}$$
$$\tilde{\tau}_{22} = M\tilde{\gamma}_{22}$$
$$\tilde{m}_{32} = 4GR^2\tilde{\kappa}_{32}$$

where G_c is the Cosserat shear modulus.

The momentum balance equations become:

$$\frac{\partial \tau_{12}}{\partial x_2} = \rho \ddot{u}_1; \quad \frac{\partial \tau_{22}}{\partial x_2} = \rho \ddot{u}_2$$
$$\frac{\partial m_{32}}{\partial x_2} + \tau_{21} - \tau_{12} = I \ddot{\omega}_3^c.$$

At steady state we have a Cauchy continuum under homogeneous shear: $\tau_{(12)} = \tau^*_{(12)} = \tau_0$, $\tau_{22} = \tau^*_{22} = \sigma_0$, $\tau_{[12]} = \tau^*_{[12]} = 0$ and $m_{32} = m^*_{32} = 0$

Perturbations: $u_i = u_i^* + \tilde{u}_i$, $\omega_3 = \omega_3^{c*} + \tilde{\omega}_3^c$

Replacing:

$$\frac{\partial \tilde{\tau}_{12}}{\partial x_2} = \rho \ddot{\tilde{u}}_1; \quad \frac{\partial \tilde{\tau}_{22}}{\partial x_2} = \rho \ddot{\tilde{u}}_2$$
$$\frac{\partial \tilde{m}_{32}}{\partial x_2} + \tilde{\tau}_{21} - \tilde{\tau}_{12} = I \ddot{\tilde{\omega}}_3^c.$$

Solution:

$$\tilde{u}_i = U_i e^{st+\mathrm{i}kx}$$
$$\tilde{\omega}_i^c = \Omega_i e^{st+\mathrm{i}kx}$$

with $k=\frac{2\pi}{\lambda}$ satisfying the BC's:

$$\tilde{\sigma}_{12}\left(x_2=\pm\frac{H}{2}\right)=\tilde{\sigma}_{22}\left(x_2=\pm\frac{H}{2}\right)=\tilde{m}_{32}\left(x_2=\pm\frac{H}{2}\right)=0$$

Replacing and solving for s yields:

$$s=\mathrm{i}kv_p \quad \text{or}$$
$$s=\pm\mathrm{i}kv_s\sqrt{\frac{h}{h+1}}\sqrt{\frac{\eta_1\left(1+\frac{1}{k^2R^2}\right)+\frac{h+1}{h}}{\frac{\eta_1}{k^2R^2}+1}},$$

where $I=0$ for simplicity.

The system is unstable when $Re[s] > 0$:

$$h < 0 \text{ (softening) and } \eta_1 \left(1 + \frac{1}{k^2 R^2}\right) + \frac{h+1}{h} > 0$$

or

$$\lambda > \lambda_{cr} = 2\pi R \sqrt{-\frac{1+h}{h} - \frac{1}{\eta_1}}$$

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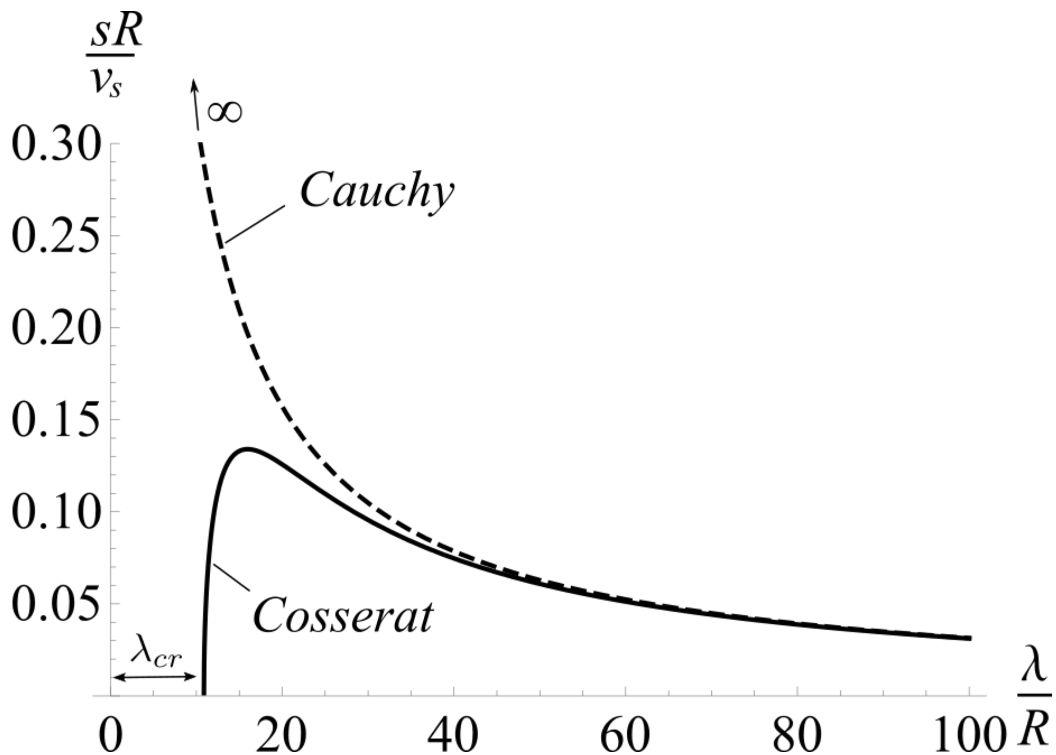
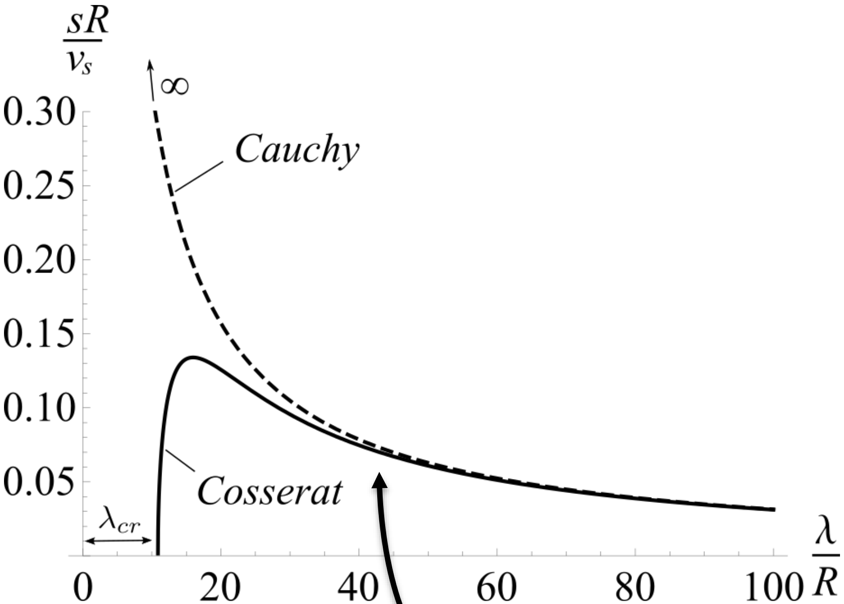
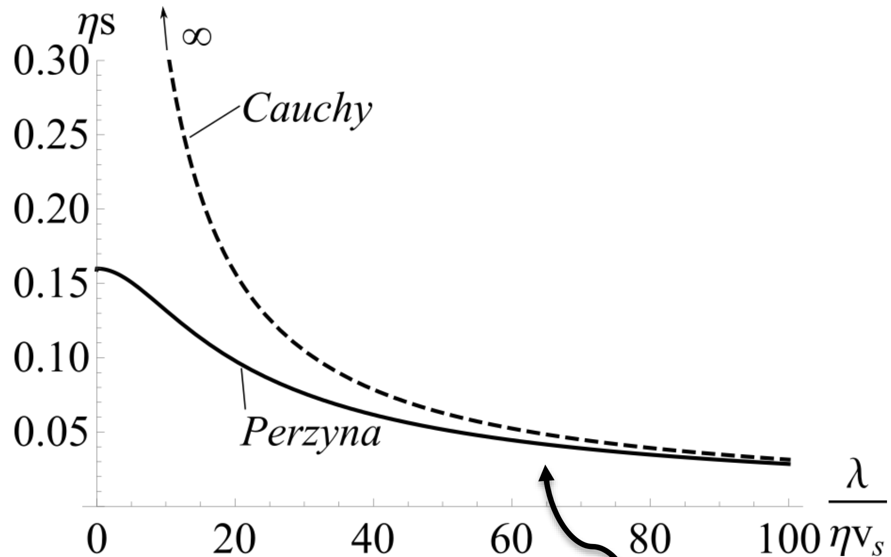


Cosserat vs Viscoplasticity





$$s = -v_s^2 \frac{\eta f_0}{2G(1+h)^2} k^2 + \sqrt{\left(v_s^2 \frac{\eta f_0}{2G(1+h)^2} k^2\right)^2 - v_s^2 \frac{h}{1+h} k^2}$$

$$s = +\mathrm{i} k v_s \sqrt{\frac{h}{h+1}} \sqrt{\frac{\eta_1\left(1+\frac{1}{k^2R^2}\right)+\frac{h+1}{h}}{\frac{\eta_1}{k^2R^2}+1}}$$

Multiphysics couplings (characteristic time & length)

Exercise #3: Cauchy layer with 2-way (strong) thermo-mechanical coupling

Linearized constitutive law:

$$\tilde{\sigma}_{12} = 2G\frac{h}{1+h}\tilde{\varepsilon}_{12} + A\tilde{T}$$

Heat equation (perturbed):

$$\frac{\partial \tilde{T}}{\partial t} = c_{th}\frac{\partial^2 \tilde{T}}{\partial x^2} + 2\tau^*\tilde{\varepsilon}_{12},$$

c_{th} is the thermal diffusivity

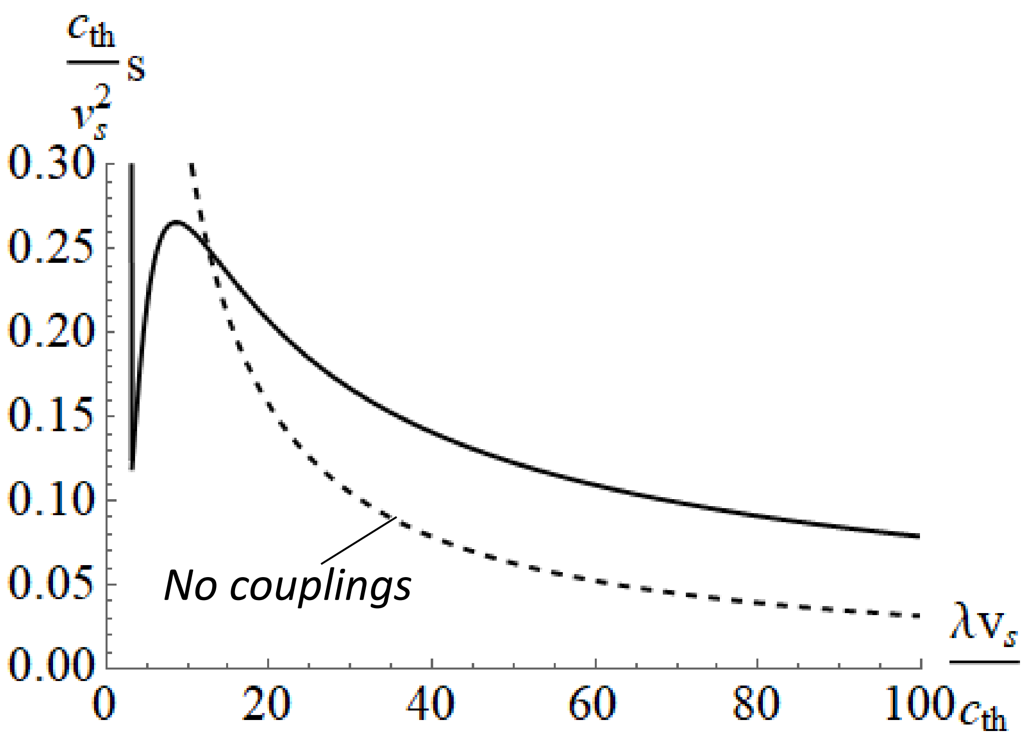
τ^* the shear stress at the state of homogenous deformation.

From balance equation we obtain:

$$-(k^2 v_s^2 \frac{h}{1+h} + s^2)g + \mathrm{i}k\frac{A}{\rho}\theta = 0$$

From heat equation:

$$\mathrm{i}k\tau^* g - \left(k^2 c_{th} + s\right)\theta = 0$$



Summing up…

- Bifurcation analysis leads to conditions for strain localization under different constitutive assumptions, continua and multiphysics couplings;
- Scaling helps to identify the dominant time and spatial scales;
- Deformation bands are a type of strain localization, commonly met
- Linear stability analysis gives the band's thickness and mesh dependency without cumbersome numerical analyses;
- We showed analytically why mesh dependency takes place;
- Regularization techniques restore physics and alleviate mathematical artifacts, such as instantaneous localization on a mathematical plane.
- Viscosity → characteristic time;
- Micromorphic continua → characteristic length;
- Multiphysics → characteristic length & time.

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Diffuse bifurcation

Thank you for your attention!

References

References

See chapter…