Definition and Uses of the Principle of Virtual Power

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The Principle of Virtual Power (PVP) offers a systematic approach for studying the equilibrium of complex systems. This chapter aims at showing the importance of the principle and its use in discrete and continuum systems. Simple examples are given throughout the chapter for helping understanding. The general theory of micromorphic continua is also derived using the principle of virtual power. Finally, PVP being global rather than local, is directly amenable to numerical schemes such as the Finite Element Method (FEM). An example is given using the open-source FEM library FEniCS.

1 Introduction

The Principle of Virtual Power (PVP) offers a systematic approach for studying the equilibrium of complex systems following an energy- and velocity-based approach. PVP shows an important advantage compared to the conventional approach of force equilibrium equations as it is simpler to apply in complex systems. Moreover, forces play a secondary role in PVP as they appear only as conjugate in energy quantities to velocities, which are in principle measurable.

After studying this chapter, the reader will be able to understand the importance of the principle of virtual power, its use in problems of discrete systems and continuum mechanics and its application to the Finite Element Method (FEM). Simple examples are given throughout the chapter for helping understanding.

The current chapter has the following structure. In the beginning of section 2 the statement of the principle is given, accompanied by a brief historical note, in order to clarify the fundamental ideas of the principle and its connection with the equilibrium equations. Examples are then given for showing the application of PVP to simple problems involving discrete systems of one and
several degrees of freedom. In section 3, the principle is extended to continuum mechanics, where the equivalence between the strong form of the differential equilibrium equations and the weak/variational form of the problem is shown. Further, in section 4, the principle is used for deriving more advanced continuum theories, i.e. the micromorphic continuum theory. The equations of micromorphic continua are presented in a general form and the hierarchical structure of the theory is illustrated. Cosserat and strain gradient theories are presented as special cases. Applications are also presented for showing the advantages of micromorphic continua and the use of PVP in upscaling. Finally, the use of the principle in the Finite Element Method is shown and the example of shearing of an infinite layer modeled as a Cauchy and a Cosserat continuum is presented, using the FEniCS open source FEM library.

2 The Principle of Virtual Power

2.1 Statement

There is no doubt that the intuition for the Principle of Virtual Power (PVP) existed from ancient times. However, the principle is used in a more systematic time in the 18th century. It is difficult to say who is the father of the principle, maybe Galileo Galilei, René Descartes, Jean-Baptiste le Rond d’Alembert or Johann Bernoulli. The first, clear, general and direct statement of the principle seems to be by Joseph-Louis Lagrange in 1788 in his seminal work. The principle reads [Lag88; p.10-11]:

“It is worth noting that Lagrange uses the term “puissance” (power in English) instead of the term “force”, which is commonly used today. The word “puissance” comes from the old French verb “pouvoir”, which means “be able to”. In this sense a force is a quantity that enables motion, the cause.

1Translation in English by G.A. Maugin [Mau14]: If any system of bodies or points as we want, is acted upon by any system of powers, is in equilibrium, and we give to this system any small motion, then by virtue of the fact that each point travels an infinitesimally small space that expresses its virtual velocity, the sum over powers each multiplied by the space that the point where it is applied travels along the direction of the same power, will always be equal to zero, regarding as positive the small distances followed in the direction of the powers and as negative those travelled in the opposite direction.
Another central notion in the above principle is the notion of virtual velocities. According to Lagrange \(^{\text{Lag88}}\):

"On doit entendre par vitesse virtuelle, celle qu’un corps en équilibre est disposé à recevoir, en cas que l’équilibre vient d’être rompu ; c’est-à-dire la vitesse que le corps prendroit réellement dans le premier instant de son mouvement."

Virtual velocities are “virtual” and not “real”, in the sense that they are possible velocities that could be developed, if the equilibrium is not anymore satisfied. Virtual velocities should be thought as variations from the reference/equilibrium state.

It is worth mentioning that the principles of virtual power (velocities) and virtual work (displacements) are practically the same. However, there is a small difference. The principle of virtual power has the advantage to be applied without considering infinitesimal displacements and rotations.

2.2 PVP applied on a single body

In the case of only one body, the principle says that the body will be in equilibrium if, and only if, the power generated by the forces acting on it is null under any possible (virtual) velocity of the body. The power is said to be “virtual”, because there is no need for the particle to actually move to apply the principle. It only needs to be in equilibrium (steady state), either moving, or at rest.

![Figure 1: A rigid body under the action of three forces and virtual velocities.](image-url)
Take for example the undeformable solid of Figure 1, where three forces are applied. The body will be in equilibrium if, and only if, the total power of the applied forces \( F^{(i)} \) is zero for any kinematically admissible virtual velocity \( \mathbf{v}^{(i)} \), i.e.:
\[
\sum_i \mathbf{P}^{(i)} = \sum_i \mathbf{F}^{(i)} \cdot \mathbf{v}^{(i)} = 0, \quad \forall \mathbf{v}^{(i)}.
\]

(1)

The fact that the solid is not deformable imposes the following constraint to the virtual velocities:
\[
\mathbf{v}^{(i)} = \mathbf{v}^O + \mathbf{\omega}^O \times \mathbf{r}^{(O,i)},
\]

(2)

where \( \mathbf{v}^O \) and \( \mathbf{\omega}^O \) represent, respectively, virtual translational and rotational velocities with respect to an arbitrary point \( O \) and \( \mathbf{r}^{(O,i)} \) is the position vector of the application point of force \( i \) with respect to \( O \). The virtual velocities, \( \mathbf{v}^{(i)} \), have to respect equation (2) or any other restrictions (e.g. boundary conditions), i.e. to be kinematically admissible. Using equation (2), equation (1) becomes:
\[
\sum_i \mathbf{F}^{(i)} \cdot \mathbf{v}^O + \sum_i \mathbf{r}^{(O,i)} \times \mathbf{F}^{(i)} \cdot \mathbf{\omega}^O = 0, \quad \forall \mathbf{v}^O, \mathbf{\omega}^O.
\]

(3)

As this equation is valid for any virtual displacement and rotation, we can derive (deduce) the standard force and moment equilibrium equations:
\[
\sum_i \mathbf{F}^{(i)} = 0
\]
\[
\sum_i \mathbf{M}^{(O,i)} = 0
\]

(4)

where \( \mathbf{M}^{(O,i)} = \mathbf{r}^{(O,i)} \times \mathbf{F}^{(i)} \) is the moment of force \( i \) w.r.t. point \( O \).

Notice that the reverse procedure is also possible. Starting from the equilibrium equations, equations (4), multiplying each one by arbitrary quantities \( \mathbf{v}^O \) and \( \mathbf{\omega}^O \), adding them and using equation (2) we retrieve equation (1). This shows the equivalence between the principle of virtual power and the equilibrium equations.

2.3 Generalized forces

The notion of power in the PVP allows the consideration of generalized forces such as moments (also called double forces, dipoles, couples) or even more general quantities (e.g. triple forces, tripoles, see section 4), which are conjugate in energy with generalized velocities. For instance, in the case of a moment, \( \mathbf{M} \), the generalized velocity is an angular velocity \( \mathbf{\omega} \). In the case of a system with \( j \) external forces and \( k \) external moments, equation (1) becomes \((i = j + k)\):
\[
\sum_i \mathbf{P}^{(i)} = \sum_j \mathbf{F}^{(j)} \cdot \mathbf{v}^{(j)} + \sum_k \mathbf{M}^{(k)} \cdot \mathbf{\omega}^{(k)} = 0, \quad \forall \mathbf{v}^{(j)}, \mathbf{\omega}^{(k)}.
\]

(5)
Following the same procedure as the previous paragraph, the standard equilibrium equations are retrieved:

\[
\sum_j F^{(j)} = 0 \\
\sum_j M^{(O,j)} + \sum_k M^{(k)} = 0
\]  

(6)

where \( M^{(O,j)} = r^{(O,j)} \times F^{(j)} \) is the moment of the external force \( j \), \( F^{(j)} \), w.r.t. point \( O \), while \( M^{(k)} \) is a concentrated external moment.

### 2.4 PVP applied on a system of bodies

Consider now a system of several bodies that interact one with another through a system of internal forces \( F^{(int,i)} \). These internal forces oppose the relative movement of the bodies (think of a spring for example that connects two bodies). Consequently, according to the PVP, their virtual power has to be considered with a negative sign. On the same system of bodies, we apply also external forces, \( F^{(ext,i)} \). PVP says that the system will be in equilibrium if, and only if, the power of (all) the forces acting on it (both external and internal) is zero, under any virtual velocity:

\[
\sum_i \bar{P}^{(i)} = \sum_j \bar{P}^{(ext,j)} - \sum_k \bar{P}^{(int,k)} = 0, \quad \forall \bar{P}^{(i)}.
\]  

(7)

Example: Consider the beam of length, \( L \), presented in figure \( 2 \). Both the horizontal and vertical displacements are fixed at point A, while only the vertical displacement is fixed at point B. The system is statically determinate. A vertical external force is applied in the middle of the beam as shown in figure \( 2 \). For calculating the internal moment at point D, we remove the kinematical constraint that assures the continuity of the system at that point and we replace it with a system of internal moments \( M^{int,left}_D = M^{int,right}_D = M^{int}_D \). The PVP (equation (7)) yields:

\[
F \delta_C - \left( M^{int,left}_D \bar{\alpha} + M^{int,right}_D \bar{\beta} \right) = 0 \quad \forall \delta_C, \bar{\alpha}, \bar{\beta}.
\]  

(8)

From geometrical compatibility we obtain (see figure \( 3 \)):

\[
\delta_D = \bar{\alpha} \frac{L}{4} = \bar{\beta} \frac{3L}{4} \quad \text{and} \quad \delta_C = \bar{\beta} \frac{L}{2}.
\]  

(9)

Replacing equation (9) to equation (8) we finally obtain that \( M^{int}_D = \frac{FL}{8} \).

### 3 The principle of virtual power in continuum mechanics

The above concepts are extended in continuum mechanics, provided that the various kinematic and stress fields show a certain mathematical regularity
The first step is to define the internal and external virtual powers. Consider a solid of volume $V$, with boundary $S$, as shown in figure 3. At $S_v$ velocities are prescribed, $v_i = v_i^{S_v}$, while at $S_p$ tractions are imposed, $t_i = t_i^{S_p}$. $S = S_v \cup S_p$ and $S_v \cap S_p = \emptyset$. Body forces, $f_i$, are applied everywhere on the solid. In the rest of this manuscript, indicial notation is followed using Einstein’s notation, i.e. repeated indices denote summation. In three dimensions $i = 1, 2, 3$. 

Assuming the following forms for the internal and external virtual power densities:
\[
\begin{align*}
\dot{\tilde{\rho}}^{(int)} &= \sigma_{ij}\tilde{\epsilon}_{ij} \\
\dot{\tilde{\rho}}^{(ext,t)} &= t_i\tilde{v}_i \\
\dot{\tilde{\rho}}^{(ext,f)} &= f_i\tilde{v}_i,
\end{align*}
\]  
\tag{10}

where \(\tilde{\epsilon}_{ij} = \frac{1}{2} (\tilde{e}_{ij, j} + \tilde{e}_{ij, i})\) and \(\sigma_{ij} = \sigma_{ji}\), according to equation (7) and paragraph 2.1, the principle of virtual power says that any part of the system will be in equilibrium, if and only if for any sub-volume, \(D\), of \(V\) with boundary, \(\partial D\), the following equality is satisfied:

\[
\int_D \sigma_{ij}\tilde{\epsilon}_{ij}dV - \int_D f_i\tilde{v}_i dV - \int_{\partial D} t_i\tilde{v}_i dS = 0,
\]  
\tag{11}

for any kinematically admissible virtual velocity field \(v_i\), i.e. \(\forall v_i\) with \(v_i = v^{S,v}_i\) on \(S_v\).

Applying the divergence theorem and after some algebra, we obtain:

\[
\int_D (\sigma_{ij,j} + f_i)\tilde{v}_i dV + \int_{\partial D} (t_i - \sigma_{ij}n_j)\tilde{v}_i dS = 0,
\]  
\tag{12}

where \(n_i\) is the normal vector to \(S\).

The above equation holds for any volume \(D \subset V\) and \(\forall v_i\) and therefore it has to hold:

\[
\sigma_{ij,j} + f_i = 0 \quad \text{and} \quad t_i = \sigma_{ij}n_j.
\]  
\tag{13}

The above equations form the standard equilibrium equations and stress boundary conditions in the classical, Cauchy continuum. As in the simpler case of discrete forces, the reverse procedure is possible, showing the full equivalence of the principle of virtual power (weak form) with the differential equilibrium equations (strong form). We say that equation (12) is the weak formulation of the strong form, equations (13), of the problem. The weak form of a Partial Differential Equation (PDE) is called also variational form and it is the starting point for obtaining numerical solutions with the Finite Element method. For an introduction to variational calculus and applications we refer to Fun65.

Nowadays, several Finite Element codes exist that allow the user to automatically and efficiently solve PDE’s in parallel by directly entering in symbolic, mathematical form the variational form of the problem (e.g. Fenics project [ABH15], GetDP [GDL98], FreeFEM++ [Hec12], among others). An example is given in section 5.

Applying one more time the divergence theorem on the whole solid of volume \(V\), we obtain:

\[
\int_{\partial V} t_idS + \int_V f_idV = 0.
\]  
\tag{14}
Multiplying equations (13) with $\epsilon_{ijk} x_k$, where $\epsilon_{ijk}$ is the Levi-Civita symbol, and using once more the divergence theorem, we obtain the following equation:

$$
\int_{\partial V} \epsilon_{ijk} t_j x_k dS + \int_V \epsilon_{ijk} f_j x_k dV = 0,
$$

which represents a cross product (moment). Equations (14) and (15) are called, respectively, linear and angular momentum balance equations. They are the analogue of equations (4) in continuum mechanics.

4 Micromorphic continua and the method of virtual power in continuum mechanics

In the classical continuum theory, the material point is characterized by its position and velocity. This description is abandoned in non-Newtonian, quantum physics, where it is described by a single quantity, the wave function. Staying in the Newtonian context there are various situations where one might need to assign more than translational degrees of freedom to the material point, seen now as a particle.

A more general continuum theory that, by construction, can indeed account for an arbitrary number of degrees of freedom assigned at the material point is the Micromorphic theory. This theory is general enough to represent various heterogeneous systems with microstructure of non-negligible size and take into account various length and time scales (internal lengths) that the classical Cauchy continuum fails to represent. The various features of the Micromorphic continuum theory were studied by many researchers in the past, showing several advantages compared to the classical continuum approach. Intrinsic wave dispersion, regularization in strain localization problems, non-singular fields in fracture mechanics, interesting properties related to the design of metamaterials, are some of the applications that emerge from the deep study of these continua.

According to Germain [Ger73], in the classical description, a continuum is a continuous distribution of particles, each of them being represented geometrically by a point and characterized kinematically by a velocity $V_i$. In a theory that takes microstructure into account, from the macroscopic point of view (which is the point of view of a continuum theory), each particle is still represented by a point $M$, but its kinematical properties are defined in a more detailed way.

At the microscopic level of observation, a particle appears itself as a continuum $P(M)$ of small extent. Let $M$ be the center of mass of the particle $P(M)$, $M'$ a point of $P(M)$, $V_j$ the displacement of $M$, $x'_i$ the coordinates of $M'$ in a Cartesian frame parallel to the given, global frame and $M$ its origin, $V'_i$ the velocity of $M'$ with respect to the given frame and $x_i$ the coordinates of $M$
in the given frame (see figure 4). \( D \) denotes the control volume. As \( P(M) \) is of small extent, it is natural to look at the asymptotic expansion of \( V'_i' \) with respect to \( x'_i' \):

\[
V'_i' = V'_i + \chi_{ij} x'_j + \chi_{ijkl} x'_j x'_k + \chi_{ijklm} x'_j x'_k x'_l + \ldots,
\]

(16)

where \( \chi_{ij} \) is a micro-deformation rate tensor, which expresses the gradient of the relative velocities \( V'_i' \) and \( \chi_{ij...m} \) are higher order micro-deformation rate tensors. In three dimensions: \( i, j, \ldots, m = 1, 2, 3 \). The tensors \( \chi_{ij...m} \) are assumed to be fully symmetric with respect to the indices \( j, \ldots, m \).

The virtual power density of the internal forces for a micromorphic continuum of order \( n \) is given as follows by [Ger73]:

\[
\tilde{p}^{int} = \tau_{ij} \tilde{V}_{i,j} - (s_{ij} \tilde{\chi}_{ij} + s_{ijk} \tilde{\chi}_{ijk} + \ldots) + (\nu_{ij} \tilde{\kappa}_{ij} + \nu_{ijkl} \tilde{\kappa}_{ijkl} + \ldots),
\]

(17)

with \( \tau_{ij} \equiv \sigma_{ij} + s_{ij} \), where \( \tau_{ij} \) is the stress tensor, \( \sigma_{ij} \) is the intrinsic stress tensor (symmetric due to objectivity requirement), \( s_{ij} \) is the intrinsic microstress tensor, \( \nu_{ij} \) is the intrinsic second microstress tensor and \( s_{ij...m} \), \( \nu_{ij...ml} \) are higher order stress tensors that are conjugate in energy to \( \chi_{ij...m} \) and \( \kappa_{ij...ml} = \chi_{ij...m,l} \), respectively. \( \cdot' \) denotes derivation in terms of \( x_i \) (macro-coordinate). The virtual power density of the external forces for a micromorphic continuum of order \( n \) is given as follows [Ger73]:

\[
\tilde{p}^{(ext,t)} = f_i \tilde{v}_i + \psi_{ij} \tilde{\chi}_{ij} + \psi_{ijk} \tilde{\chi}_{ijk} + \ldots \]

\[
\tilde{p}^{(ext,f)} = f_i \tilde{v}_i + \psi_{ij} \tilde{\chi}_{ij} + \psi_{ijk} \tilde{\chi}_{ijk} + \ldots,
\]

(18)
where $f_i$, $\psi_{ij...l}$ represent volumic (body) generalized forces and $t_i$, $\mu_{ij...l}$ generalized tractions. In particular, $t_i$ is the surface traction, $\mu_{ij}$ is the double surface traction (dipole, e.g. a concentrated moment density) and $\mu_{ij...n}$ is the generalized surface traction of order $n$ defined on the boundary $\partial V$. Similarly, $f_i$ represent long-range volumic forces, $\psi_{ij}$ double long-range volumic forces (e.g. due to an electromagnetic field) and $\psi_{ij...n}$ higher order generalized volumic forces of order $n$ defined on $V$.

Applying the principle of virtual power and using the divergence theorem, we obtain \cite{Ger73}:

\[
\tau_{ij,j} + f_i = 0, \quad t_i = \tau_{ij} n_j
\]
\[
\nu_{ijk,k} + s_{ij} + \psi_{ij} = 0, \quad \mu_{ij} = \nu_{ijk} n_k
\]
\[
\nu_{ijkl,l} + s_{ijk} + \psi_{ijk} = 0, \quad \mu_{ijk} = \nu_{ijkl} n_l
\]
\ldots
\]

where, again, $n_i$ is the outward pointing unit normal vector field of the boundary. The above system of equations represents the equilibrium equations of a micromorphic continuum of order $n$ (strong form).

It is worth emphasizing that the above equations are derived systematically without any particular hypotheses besides the postulates of the internal and external power densities. No assumption was also made regarding the constitutive behavior of the system, i.e. the relation of the generalized stresses with the generalized deformations. The above approach can be easily generalized to take into account inertial effects \cite{Ger73}. In this case, the additional degrees of freedom of micromorphic continua introduce microinertia terms, whose presence leads to interesting wave dispersion properties, especially at short wavelengths (optic branch) \cite{SSV10} and finite Lyapunov exponents in localization problems \cite{SSV11}.

Without doubt that micromorphic theory is rich enough to fit various physical situations. This is a strong point of the theory, but it is also its weak point. The discovery of the practical significance of some of the concepts which have been introduced (e.g. of the higher order terms), the design of a general method for exhibiting their physical validity (if any), and their measurement in some specific physical situations is not obvious \cite{Ger73} and remains an open research topic. The situation, though, becomes more tractable in some special cases of 1st order micromorphic continua, where applications exist in various disciplines.

### 4.1 Special cases of micromorphic continua

In figure \ref{fig:micromorphic-theories} we outline the various higher order (micromorphic) continuum theories and their special cases. Besides the classical continuum and the Cosserat continuum (called also micropolar continuum, see \cite{Var09}), a special case of micromorphic continuum is also the second gradient and the indeterminate couple stress theory (called also restrained Cosserat medium).
Figure 5: Higher order continuum theories according to Germain’s terminology [Ger73]; see also [Min64, Eri99].
Retrieving the classical, Boltzmann continua, is straightforward by setting $\chi_{ij}$ and the higher order microdeformation rate tensors null. In this case, $s_{ij} = 0$ and $\tau_{ij} = \sigma_{ij}$, i.e. equal to the Cauchy stress tensor, which is symmetric.

In the case that the particle $P(M)$ is deformable and its microdeformation coincides with the deformation of the (macro-)continuum, i.e. $\chi_{ij} = V_{ij}$, we obtain the so-called second gradient continuum theory. As in this case the microdeformation rate tensor is no more an independent generalized virtual velocity, one has to start from the very beginning and apply the principle of virtual power for deriving the strong form of the equilibrium equations and the appropriate boundary conditions. For more details we refer to [Ger73] and for some interesting applications of the theory to [DSMP93], [CCE98], [ZPV01], [CCC06], [KABC08], [PZ16], [DAD+17], among others.

4.2 The Cosserat continuum

The derivation of the Cosserat continuum is more direct than the second gradient. The basic assumption is that the particle $P(M)$ behaves as a rigid body and so it can not only translate, but also rotate. In this case the microdeformation rate tensor has to be anti-symmetric and the rest higher-order microdeformation tensors zero.

The Cosserat continuum is easily retrieved by setting (hypothesis of rigid particle) $\chi_{ij} = -\epsilon_{ijk}\omega^k$, $k_{ij} = \omega^c_{ij}$, $s_{ij} = -\frac{1}{2}\epsilon_{ijk}s_k$, $\mu_{ij} = -\frac{1}{2}\epsilon_{ijk}\mu_k$, $\psi_{ij} = -\frac{1}{2}\epsilon_{ijk}\psi_k$ and using $\tau_{ij} = \sigma_{ij} + s_{ij}$ (see equation (17)) and the property $\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$, where $\delta_{ij}$ is the Kronecker delta, equations (17) and (18) become:

$$\tilde{p}_{int} = \tau_{ij}\tilde{\gamma}_{ij} + m_{ij}\tilde{k}_{ij}$$
$$\tilde{p}^{(ext,t)} = t_i\tilde{v}_i + \mu_i\tilde{\omega}^c_i$$
$$\tilde{p}^{(ext,f)} = f_i\tilde{v}_i + \psi_i\tilde{\omega}^c_i$$

where $\gamma_{ij} = u_{ij} + \epsilon_{ijk}\omega^k$.

The equilibrium equations (equations (19)) take the following form:

$$\tau_{ij,j} + f_i = 0, \quad t_i = \tau_{ij}n_j$$
$$m_{ij,j} - \epsilon_{ijk}\tau_{jk} + \psi_i = 0, \quad \mu_i = m_{ij}n_j$$

This is the strong form of the Cosserat continuum equations. In figure 6 the stresses and couple-stresses (moments) of a Cosserat continuum are illustrated. It is worth emphasizing that the derivation is based on the principle of virtual power and not on the linear and angular momentum balance equations. These momentum balance equations can be deduced by integrating equation (21) over the volume $V$ and by applying the divergence theorem, as in the case of the classical Cauchy continuum presented in section 3. Consequently, the method of PVP enables us to derive in a safe and systematic way complex, higher-order balance equations, that physical intuition can hardly bring us to.
4.3 PVP and upscaling for deriving constitutive laws

The above continua cannot be used unless appropriate constitutive laws are used for solving engineering problems. Of particular interest are heterogeneous systems, where the higher-order continuum theories show several advantages (e.g., physically-based regularization in strain localization problems and wave dispersion). Constitutive models can be derived either experimentally, by trial and error, or by explicitly considering the microstructure using upscaling techniques.

Upscaling (or homogenization) is a class of methods that aim at deriving an equivalent continuum theory that describes the macroscopic behavior of heterogeneous systems. (Asymptotic) Homogenization is a mathematically rigorous, well established theory for performing this task [BP89, SP86, SP88, PdCOTD09, Cha10, CN84, TC97, ABG09]. This method is based on the asymptotic expansion of the various state fields (displacements, deformations, stresses) in terms of a small quantity $\varepsilon$, which represents the ratio of the characteristic size of the elementary volume over the overall size of the structure, and provides an equivalent to the heterogeneous system homogeneous continuum for $\varepsilon \to 0$. Besides the rigorous mathematical formulation of this approach, its main advantage is the ability to determine error estimators of the derived continuum for finite values of $\varepsilon$. However, when it comes to generalized continua, such as the Cosserat continuum, that possess internal lengths, the asymptotic limit $\varepsilon \to 0$ looses interest as it cancels out the internal lengths [FS98, FPS01]. In other words, by this pass to the limit, asymptotic homogenization erases any
internal lengths related to the material’s micro-structure, which higher order continuum theories, such as Cosserat are in principle able to capture. The separation of scales, intrinsic in asymptotic homogenization theory, does not hold anymore.

To overcome this problem several alternative schemes have been proposed for upscaling heterogeneous systems (see for example [AL94, FS98, BV01, SSV08, SSV10, BG12, GSS16, RC16], among others). The majority of these schemes is based upon the “homogeneous equivalent continuum” concept [Cha10], in the sense that the derived higher order continuum shares a) the same power (internal) and b) the same kinematics with the heterogeneous medium for any generalized virtual velocity field. This approach reminds us the PVP, applied on a reduced space of kinematics and averaging. The classical asymptotic homogenization expansion Ansatz that leads to a Cauchy continuum as the ratio of the size of the unit cell over the overall structure tends to zero is not followed in this case. Therefore, these heuristic approaches remain applicable even when the size of the microstructure is not infinitesimal as compared to the overall size of the system, or in other words, when scale-separation is no satisfied.

A typical example for applying and testing these upscaling methods are masonry-like structures. Masonry can be seen as a geomaterial whose building blocks are often quasi-periodically arranged in space. Moreover, the building blocks are at the human scale, which makes them an ideal toy-model, contrary to granular media whose microstructure is small, shows topological complexity and has to be statistically described. When the upscaling scheme is correctly formulated, it is possible to capture the wave dispersion behavior of a heterogeneous system, even when the wave length is comparable to the block size. Notice that in this case, the classical, Cauchy continuum approach fails as it is not a dispersive medium. In figure 7 we present the modal frequencies of a masonry panel that was upscaled with Cosserat continuum versus the number of its building blocks. Even when the number of the building blocks is small, the Cosserat homogenized continuum model succeeds in representing the dynamics of the discrete heterogeneous structure. In figure 8 we present the out-of-plane-displacement contours of the first three flexural modal shapes of a homogenized masonry panel and the comparison with the flexural modal shapes provided by the Discrete Element Method (DEM) [GSS14]. It is worth mentioning that DEM provides very satisfactory results compared to well controlled experimental tests and, therefore, it can be used as reference [GSS17]. The upscaled Cosserat continuum behaves very well, even for non-linear material behavior (see figure 9).

Nonlinear behavior, accompanied by strain softening is inherent in granular materials [Var09]. In figure 10 we show the response of a granular layer under shearing studied with the Discrete Element Method. Upscaling granular media and transferring adequate information at the macroscale (e.g. inter-
Figure 7: Modal frequencies of a masonry panel versus the number of building blocks: comparison between the results extracted by the Discrete Element Method and by the use of the Cosserat continuum [SSV08, GSS13].

Figure 8: Out-of-plane-displacement contours of the first three flexural modal shapes. Left: Discrete Elements solution. Right: Cosserat Finite Element solution [GSS14].
Figure 9: Numerical simulation of a confined masonry panel undergoing shear deformation. Above: tested configuration (left) and normalized force-displacement curves from DEM and Cosserat FEM analyses (right); Below: Comparison between the pattern of plastic deformation obtained from the discrete (left) and Cosserat Finite Element (right) models [GSSS16].
nal lengths) is a challenging topic \cite{BV01, RC16, GSSS16} and can provide valuable information on strain localization and energy dissipation in the absence of detailed experimental data, which is often the case for fault gouges due to their complex thermo-hydro-chemical behavior (see Figure \ref{fig:12}, \cite{SSV11, VSS13, SS16, RSS18a, RSS18b}). However, Cosserat continuum is effective under shearing. In the case of pronounced extension or compaction, Cosserat kinematics have no effect and at least a complete first order micromorphic continuum has to be used or, alternatively, a second gradient model (restrained 1st order micromorphic).

Figure 10: Example of DEM simulation of a granular layer under shearing with constant velocity and shear stress-strain response (courtesy: Efthymios Papachristos).

5 Finite elements

The principle of virtual power, being global rather than local, is directly amenable to numerical schemes such as the Finite Element Method (FEM). Moreover, it is independent of constitutive laws, providing high degree of generality and abstraction. In this paragraph, we give two simple examples showing its direct use with the Finite Element library FEniCS \cite{ABH15}. We don’t get into the details of the finite element method. For a consistent presentation of the method, the interested reader is referred to classical textbooks such as those of \cite{ZT13, Hug00, Bat07}.

The problem solved is that of the infinite layer of height $h$ subjected to shearing, as shown in figure \ref{fig:12}. Half of the layer is modeled. Both a Cauchy and

\footnotesize

\begin{itemize}
  \item Installing FEniCS is straightforward in all operating systems: \url{https://fenicsproject.org}.
  \item In Windows we suggest installing the Windows subsystem for Linux: \url{https://docs.microsoft.com/en-us/windows/wsl/about}.
\end{itemize}
Figure 11: Apparent rate dependency of an infinite layer of a fault gouge under shearing ($\frac{\partial \mathbf{u}}{\partial t}$), modeled as a Cosserat continuum. $h$ is the gouge thickness, often of the order of some $\mu$m. The slip rate has a direct impact on the shear stress-strain response, even in the ideal case of perfect plasticity (zero hardening) [RSS18a, RSS+18b].

A Cosserat continuum is used. Small deformations and linear elasticity is considered. The problems have analytical solutions, but their derivation and the comparison with the numerical solution is left to the reader (hint: a mesh convergence analysis should always be performed).

Figure 12: Infinite layer under shearing.
5.1 Simple shear with Cauchy continuum

Considering the invariance of the problem in $x_1$ and $x_3$ directions (infinite layer) and the symmetry of the stress tensor, $\sigma_{12} = \sigma_{21}$, equations (10) become:

$$\sigma_{22} \varepsilon_{22} + \sigma_{12} \varepsilon_{12} = \varepsilon_{\text{int}}$$

$$\sigma_{22} \varepsilon_{22} + \sigma_{12} \varepsilon_{12} = \varepsilon_{\text{ext},t}$$

with $\alpha = 1, 2$ (repeated indices denote summation). The stresses are equal to:

$$\sigma_{22} = Mu_{2,2}$$

$$\sigma_{12} = Gu_{1,2},$$

where $\nu$ is the Poisson ratio, $E$ the Young’s modulus, $G = \frac{E}{2(1+\nu)}$ the shear modulus, $M = \frac{E(1-\nu)}{4(1-2\nu)(1+\nu)}$ the P-wave modulus and $u_{\alpha}$ the displacements (unknowns of the problem).

The principle of virtual power (i.e. the variational/weak form of the problem) together with the above constitutive law and the boundary conditions are directly written in Python in symbolic/mathematical form and the library FEniCS takes on for formulating the finite element problem and solving the linear system of equations, automatically! The Python code is given below, annotated. The numerical results of this simple problem are given in figure 13.

```python
# Import FEniCS library
from dolfin import *

# Parameters
h=2.  # height of the layer
s12_bc=.5  # applied shear stress at boundary
s22_bc=.25  # applied normal stress at boundary
nu_el=0.  # Poisson coefficient
ny=3  # number of elements - FE discretization
E_el=1000.  # Young modulus
nu_el=0.  # Poisson coefficient
G_el = E_el / (2. * (1 + nu_el))
M_el = E_el * (1. - nu_el) / ((1. - 2. * nu_el) * (1. + nu_el))

# Generate mesh
mesh=IntervalMesh(ny,0.,h/2.)
# Define element topology (1D)
cell=interval
# Defines a Lagrangian FE of degree 1 and two unknowns in each node (vector)
element=VectorElement("Lagrange", cell, degree=1,dim=2)
# Assign the element to the mesh
V=FunctionSpace(mesh, element)

# Define test function (virtual velocities)
v=TestFunction(V)
# Define trial (unknown) function
u=TrialFunction(V)
# Store the solution to sol
sol=Function(V)
```
# Define boundary conditions

def middle(x, on_boundary):
    return on_boundary and near(x[0], 0.)

c = DirichletBC(V, (0., 0.), middle)

# Define traction vector

t = Constant((s22_bc, s12_bc))

# Define internal virtual power

Pint =
    M_el*Dx(u[0], 0)*Dx(v[0], 0) + #sigma_{22} v_2
    G_el*Dx(u[1], 0)*Dx(v[1], 0) #sigma_{12} v_1

# Define external virtual power

Pext = dot(t, v)*ds

# Solve the problem (does the FE formulation, matrix assembly and linear solve)
solve(Pint == Pext, sol, c)

# Plot solution

import matplotlib.pyplot as plt

plot(sol[0], label='$u_2$')
plot(sol[1], label='$u_1$')
plt.legend(loc='upper left')
plt.xlim(0, 0.001)
plt.rc('font', **font)
plt.show()

Listing 1: FEniCS Python code for Cauchy shearing.

5.2 Simple shear with Cosserat theory

Considering, again, the invariance of the problem in $x_1$ and $x_3$ directions equations (20) become:

\[
\begin{align*}
\tau_{int} &= \tau_{22} \tilde{\gamma}_{22} + \tau_{12} \tilde{\gamma}_{12} + \tau_{21} \tilde{\gamma}_{21} + m_{32} \tilde{k}_{32} \\
\rho^{(ext,t)} &= l \alpha \tilde{\nu}_a + \mu_3 \tilde{\omega}_3
\end{align*}
\]

(24)

As far as it concerns the constitutive law, the symmetric part of $\tau_{ij}$ is equal to the Cauchy stress tensor, i.e. $\tau_{(ij)} = \sigma_{ij}$, and therefore equations (23) can be used. The antisymmetric part of $\tau_{ij}$ is equal to $\tau_{(ij)} = 2G_c \gamma_{(ij)}$, where $G_c = \eta_1 G$; see \cite{Vah09}. Regarding the couple stresses, $m_{32} = M_c k_{32}$, where $M_c = 2\eta_3 G l^2$. $\eta_1$ and $\eta_3$ are material parameters and $l$ is the Cosserat internal
length. Therefore, we have:

\[
\begin{align*}
\tau_{22} &= \sigma_{22} = Mu_{2,2} \\
\tau_{12} &= (G + G_c) u_{1,2} + 2G_c\omega_c^3 \\
\tau_{21} &= (G - G_c) u_{1,2} - 2G_c\omega_c^3 \\
m_{32} &= M_c\omega_c^3
\end{align*}
\]  

(25)

Similarly to the previous paragraph, the principle of virtual power (i.e. the variational/weak form of the problem) together with the above constitutive law and the boundary conditions are directly written in Python in symbolic/mathematical form and the library FEniCS takes on the rest. Quadratic interpolation was chosen for the displacement and linear for the Cosserat rotation field. The Python code is given below and the numerical results in figure 14.

```python
# Import FEniCS library
from dolfin import *

# Parameters
h=2.  # height of the layer
s12_bc=.5  # applied shear stress at boundary
s22_bc=.25  # applied normal stress at boundary
m3_bc=.0  # applied Cosserat moment at boundary
E_el=1000.  # Young modulus
nu_el=0.  # Poisson ratio
eta_1=.8  # Cosserat coefficient for Gc
```
eta_3 = 5./2.  # Cosserat coefficient for $M_c$
lc = h/3.  # Cosserat length

ny = 10  # number of elements – FE discretization

G_el = E_el/(2.*(1 + nu_el))
M_el = E_el*(1. - nu_el)/((1. - 2.*nu_el)*(1. + nu_el))
Gc_el = eta_1*G_el
Mc_el = 2*eta_3*G_el*lc**2

# Generate mesh
mesh = IntervalMesh(ny, 0., h/2.)

# Define element topology (1D)
cell = interval

# Defines a Lagrangian FE of degree 2 for the displacements
element_disp = VectorElement("Lagrange", cell, degree=2, dim=2)
# Defines a Lagrangian FE of degree 1 for the rotations
element_rot = FiniteElement("Lagrange", cell, degree=1)

# Creates a mixed element
element = element_disp*element_rot

# Assign the element to the mesh
V = FunctionSpace(mesh, element)

# Define test functions (virtual velocities)
v = TestFunction(V)

# Define trial functions (unknown displacements and Cos. rotation)
u = TrialFunction(V)

# Store the solution to sol
sol = Function(V)

# Define boundary conditions
def middle(x, on_boundary):
    return on_boundary and near(x[0], 0.)
bcs = DirichletBC(V,(0., 0., 0.), middle)

# Define traction vector
timui = Constant((s22_bc, s12_bc, m3_bc))

# Define internal virtual power
Pint = (M_el*Dx(u[0], 0) + Dx(v[0], 0) +
        tau_22*gamma_22
        ((G_el+Gc_el)*Dx(u[1], 0) + 2*Gc_el*u[2])*(Dx(v[1], 0) + v[2]) +
        tau_12*gamma_12
        ((G_el-Gc_el)*Dx(u[1], 0) - 2*Gc_el*u[2])*(-v[2]) +
        tau_21*gamma_21
        Mc_el*Dx(u[2], 0)*Dx(v[2], 0)
        m_32*k_32)*dx

# Define external virtual power
Pext = dot(timui, v)*ds

# Solve the problem (does the FE formulation, matrix assembly and
# linear solve)
solve(Pint == Pext, sol, bcs)
Listing 2: FEniCS Python code for Cosserat shearing.

```python
# Plot solution
import matplotlib.pyplot as plt
plt.plot(sol[0], label='$u_2$')
plt.plot(sol[1], label='$u_1$')
plt.plot(-sol[2], label='${-\omega_3}$')
plt.xlabel('$x_2$')
plt.legend(loc='upper left')
plt.ylim(0, 0.001)
font = {'size': 18}
plt.rc('font', **font)
plt.show()
```

Figure 14: Calculated displacements of the sheared layer using Cosserat continuum.

6 Summary

The target of the present chapter was to give the basic ideas and intuition behind the principle of virtual power. A short historical review was made, aiming at clarifying the fundamental ideas of the principle and its connection with the equilibrium equations. After providing the statement of the principle, several examples were presented for showing its application to simple problems involving discrete systems of one and several degrees of freedom. The equivalence of the principle with the equilibrium equations was shown.
Focus was placed then on continuum systems and the generalization of the principle for deriving the differential equilibrium equations of the Cauchy continuum. The principle of the virtual power provides a systematic and rigorous way for going further and deriving the equations of more advanced continuum theories as well, i.e. the micromorphic continuum theory. The equations of the micromorphic theory were presented in a general form and the hierarchical structure of the theory was illustrated. It was shown that Cosserat and strain gradient theories are special cases of this more general framework. Some applications were also given for showing the advantages of these continuum theories and the use of the principle of virtual power for upscaling.

The principle of virtual power, being global rather than local, is directly amenable to numerical schemes such as the Finite Element Method. The application of the method provides directly the weak/variational form of the equilibrium equations, which is the starting point in any finite element formulation. Nowadays, several codes exist that allow the user performing FEM analyses by simply entering the variational form of the mathematical problem that (s)he wants to solve. One of these is FEniCS open source FEM library. The example of shearing of an infinite layer modeled as a Cauchy and a Cosserat continuum was presented for showing the FEniCS finite element implementation and the use of the principle of virtual power.

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